ZARISKI-LIKE TOPOLOGIES FOR LATTICES

BY

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This thesis is dedicated to my family; my father, my mother, my wife, my son, my daughter, my sisters and my brothers. This is also dedicated to my friends. To those who supported me along this journey. Tank you all.

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This is dedicated to my family and friends who supported me along this journey. Thank you.

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Notation

$\mathcal{L} = (L, \wedge, \vee)$	A lattice
$\mathcal{L} = (L, \wedge, \vee, 0, 1)$	A bounded lattice
$\mathcal{L}^0$	The dual lattice of $\mathcal{L}$
$Ideal(R)$	The set of all ideals of the ring R
$\mathcal{L}(M)$	The set of all submodules of M
$LAT(M)$	The lattice of submodules of M
$V(a)$	$\{x \in X \mid a \leq x\}$
$V^0(a)$	$\{y \in Y \mid y \leq a\}$
$I(A)$	$\bigwedge_{x \in A} x$
$H(A)$	$\bigvee_{x \in A} x$
$\sqrt{a}$	$\bigwedge_{x \in V(a)} x$
$\sqrt{a}^0$	$\bigvee_{x \in V^0(a)} x$
τ	$\{X \backslash V(a) \mid a \in L\}$
τ^{cl}	The classical Zariski topology
$\tau^{\mathfrak{fp}}$	The finer patch topology
$\mathcal{C}^X(L)$	The set of all radical elements with respect to X
$\mathcal{C}^X(\mathcal{L})$	The complete lattice $(\mathcal{C}^X(L), \wedge, \tilde{\vee}, \sqrt{0}, 1)$
$\mathcal{H}^Y(L)$	$\{a \in L \mid \sqrt{a}^0 = a\}$
$\mathcal{H}^Y(\mathcal{L})$	$\mathcal{C}^Y(\mathcal{L}^0)$

$Spec^p(M)$	The spectrum of prime submodules of M
$Spec^c(M)$	The spectrum of coprime submodules of M
$Spec^f(M)$	The spectrum of first submodules of M
$Spec^s(M)$	The spectrum of second submodules of M
$Spec^{fp}(M)$	The spectrum of fully prime submodules of M
$Spec^{fc}(M)$	The spectrum of fully coprime submodules of M
$\mathcal{R}(\mathcal{L})$	$\{\sqrt{x} \mid x \in L$ and $V(x)$ is irreducible $\}$
ACC (DCC)	The ascending (descending) chain condition
$SI(\mathcal{L})$	The set of all strongly irreducible elements in $\mathcal{L}$
$SH(\mathcal{L})$	The set of all strongly hollow elements in $\mathcal{L}$
$Ann(M)$	$\{a \in R \mid aM = 0\}$
$(0 :_R K)$	$\{a \in R \mid aK = 0\}$
$(0 :_M I)$	$\{x \in M \mid xI = 0\}$
$Att^s(M)$	The second attached prime ideals of M
$att^s(M)$	The main second attached prime ideals of M
A_N	$\{I \leq R \mid N \subseteq IM\}$
H_N	$Min(A_N)$
$In(N)$	$\bigcap_{I \in H_N} IM$
$Ass^h(M)$	The associated hollow ideals of M
$ass^h(M)$	The main associated hollow ideals of M
$h.dim(M)$	The hollow dimension of the module M

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

THESIS ABSTRACT

NAME:	Hamza Hroub
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The thesis consists of three chapters. In Chapter One, we study Zariski-like topologies on a proper class $X \subsetneqq L$ *of a complete lattice* $\mathcal{L} = (L, \wedge, \vee, 0, 1)$. *We consider* X *with the so called* classical Zariski topology (X, τ^{cl}) *and study its topological properties (* e.g. *the separation axioms, the connectedness, the compactness) and provide sufficient conditions for it to be spectral. We say that* $\mathcal{L}$ *is* X-top *iff*

$$\tau := \{X \backslash V(a) \mid a \in L\}, \text{ where } V(a) = \{x \in L \mid a \leq x\}$$

is a topology. We study the interplay between the algebraic properties of an X*-top complete lattice* $\mathcal{L}$ *and the topological properties of* $(X, \tau^{cl}) = (X, \tau)$. *Our results are applied to several spectra which are proper classes of* $\mathcal{L} := LAT(_RM)$ *where* M *is a left module over an arbitrary associative ring* R *(e.g. the spectra*

of prime, coprime *or* fully prime *submodules) as well as to several spectra of the dual complete lattice* $\mathcal{L}^0$ *(*e.g. *the spectra of* first, second *and* fully coprime *submodules of* M*). In Chapter Two, we work over a commutative ring* R*. We investigate* R*-modules which are* second representable*, i.e. those which can be written as finite sums of second* R*-submodules. We provide sufficient conditions for* ${}_RM$ *to be second representable, in particular within the class of lifting modules. In Chapter Three, we study firstly the complete lattice* $\mathcal{L} := (L, \wedge, \vee, 0, 1)$ *with an action from a poset* $S := (S, \leq)$ *and define several spectra related to it. In particular, we show that the spectrum* $Spec^c(\mathcal{L})$ *of coprime elements in* L *is nothing but the spectrum* $Spec^s(\mathcal{L}^0)$*, where* $\mathcal{L}^0$ *is the dual lattice with the dual action of the poset* $S^0 := (S, \geq)$*. Applying our definitions to* $S = Ideal(R)$*, where* R *is a commutative ring, and* $\mathcal{L} := LAT({}_RM)$ *where* M *is an* R*-module, we introduce a sort of dual notion to that of a primary* R*-submodule of* M*, namely that of a* pseudo strongly hollow *submodule. We provide existence and uniqueness theorems for pseudo strongly hollow representations of modules over commutative rings which generalize strongly hollow representations (which are the exact dual of the strongly irreducible representations of modules).*

ملخص الرسالة

الاسم الكامل: حمزة موسى محمد حروب

عنوان الرسالة: الفضاءات التوبولوجية الشبيهة بفضاءات زارسكي للشبكات

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نقوم في هذه الرسالة بدراسة الفضاءات التوبولوجية التي هي على شاكلة فضاءات زارسكي. في الوحدة الأولى درسنا هذه الفضاءات في إطار الشبكات التامة وهو إطار أعم وأوسع من ذلك الذي دُرست فيه بعض هذه الفضاءات سابقا. درسنا بشكل خاص الشروط التي تجعل من تلك الفضاءات التوبولوجية فضاءات زارسكي. بالإضافة إلى العديد من النتائج التي لم تكن معروفة سابقا، مكنتنا بعض النتائج التي حصلنا عليها من استرجاع بعض النتائج المعروفة حول فضاءات توبولوجية لأنواع محددة من الأطياف الخاصة بحلقيّات معرَّفة على حلقات تجميعية كحالات خاصة. قمنا أيضا بدراسة العلاقة بين الخصائص الجبرية للشبكات التامة والفضاءات التوبولوجية الشبيهة بفضاءات زارسكي المرتبطة بها. في الوحدتين الثانية والثالثة قدمنا أنواعا جديدة من التمثيلات الخاصة بالحلقيّات المعرفة على حلقات إبدالية. احتوت الرسالة على العديد من التطبيقات، كما دُعِّمت النتائج والتطبيقات بالعديد من الأمثلة.

Introduction

The spectrum $Spec(R)$ of prime ideals of a commutative ring R attains a *Zariski topology* in which the closed sets are the varieties

$$\{V(I) \mid I \in Ideal(R)\}, \text{ where } V(I) = \{P \in Spec(R) \mid I \subseteq P\}.$$

This topology is compact, T_0 but almost never T_2, and the closed points correspond to the maximal ideals. The Zariski topology proved to be very important in two main aspects: in Algebraic Geometry and in Commutative Algebra. In particular, it provided an efficient tool for studying the algebraic properties of a commutative ring R by investigating the corresponding topological properties of $Spec(R)$ [11].

Motivated by this, there were many attempts to define Zariski-like topologies on the spectra of *prime-like* submodules of a given left module M over a (not necessarily commutative) ring R. This resulted at the first place in several different notions of prime submodules of ${}_RM$ which reduced to the notion of a prime ideal for the special case $M = R$, a commutative ring (*e.g.* [46]). The work in this direction was almost limited to studying these prime-like submodules and their duals (the coprime-like submodules) as well as to the families of prime ideals corresponding to them from a purely algebraic point of view. One of the obstacles was that not every module M over a (commutative) ring R has the property that $Spec(M)$ attains a Zariski-like topology: the proposed *closed varieties* $\{V(N) \mid N \in LAT({}_RM)\}$ are not necessarily closed under finite unions. Modules for which this last condition is satisfied were investigated, among others, by R. L. McCasland and P. F. Smith (e.g. [34], [33]) and called *top modules*. However, even such modules were studied from a purely algebraic point of view and the associated Zariski-like topologies were not well studied till about a decade ago. In [6], Abuhlail introduced a Zariski-like topology on the spectrum of *fully coprime subcomodules* of a given comodule M of a coring $\mathcal{C}$ over an associative ring R and studied the interplay between the algebraic properties of M and the topological properties of that Zariski-like topology (see also [5]).

Later, in a series of papers ([3], [4], [2]), Abuhlail introduced and investigated several Zariski-like topologies for a module M over arbitrary associative ring R. These investigations showed that all the (co)prime spectra considered fall in two main classes with several common properties for the spectra in each class. Moreover, these two classes were *dual* to each other in some sense. This led Abuhlail and Lomp ([7], [1]) to investigate such topologies for a general complete lattice $\mathcal{L} := (L, \wedge, \vee, 0, 1)$ and a proper subset $X \subseteq L \backslash \{1\}$. Their main work was on characterizing the so called *X-top lattices* (i.e. $\mathcal{L}$ for which the closed varieties $V(a) := \{x \in X \mid a \leq x\}$ are closed under finite unions). In addition to the fact that this approach provides a general framework, it allowed obtaining results on the dual lattice $\mathcal{L}^0 := (L, \wedge^0, \vee^0, 0^0, 1^0) = (L, \vee, \wedge, 1, 0)$ and $X \subseteq L \backslash \{0\}$ for free.

This thesis consists of three chapters. In Chapter One, we study Zariski-like topologies for complete lattices using a different approach. Fix a complete lattice $\mathcal{L} = (L, \vee, \wedge, 1, 0)$, a subset $X \subseteq L \backslash \{1\}$ and $\tau := \{X \backslash V(a) \mid a \in L\}$.

Inspired by the work of Behboodi and Haddadi [14], [15] on the lattice $LAT(_RM)$ of submodules of a given module M over a ring R, and instead of restricting our attention to X*-top lattices* (i.e. those for which (X,τ) is a topology), we consider X with the *classical Zariski topology* (X,τ^{cl}) which is constructed on X by considering τ as a *subbase* and the *finer patch topology* $(X,\tau^{\mathfrak{fp}})$ which has a subbase $\mathcal{B} := \{V(a) \cap X\backslash V(b) \mid a,b \in L\}$. Indeed, $(X,\tau^{cl}) \leq (X,\tau^{\mathfrak{fp}})$ and $(X,\tau) = (X,\tau^{cl})$ if and only if $\mathcal{L}$ is X-top. In the special case when $\mathcal{L}$ is an X-top lattice, we not only apply the results obtained for (X,τ^{cl}), but obtain also other interesting results especially on the interplay between the algebraic properties of $\mathcal{L}$ and the topological properties of (X,τ).

In Proposition 1.22, we prove a stronger version of the converse of [1, Proposition 2.7] and conclude in Corollary 1.23 that in case $\mathcal{L}$ is an X-top lattice: $A \subseteq X$ is irreducible if and only if $I(A) := \bigwedge\limits_{x\in A} x$ is (strongly) irreducible in the sublattice $(\mathcal{C}(L),\wedge)$ of radical elements of $\mathcal{L}$. This fact was the key in the proofs of several results including Theorem 1.70 which provides 1-1 correspondences between X $(Min(X))$ and the class of irreducible sets (irreducible components) in (X,τ) provided that X contains the class $SI(\mathcal{C}(\mathcal{L}))$ of *strongly irreducible* radical elements. It is worth mentioning that Theorem 1.70 recovers several results of Abuhlail on such 1-1 correspondences for $\mathcal{L} = LAT(_RM)$ (*e.g.* [2], [3], [4]) and Abuhlail/Lomp [1] as special cases (some of these results are recovered under conditions weaker than those assumed in the original results for the different spectra of modules).

In Theorem 1.28, we prove that the class $Max(X)$ of maximal elements of X coincides with the class of $Max(\mathcal{C}(\mathcal{L}))$ of maximal radical elements. This yields, assuming that $\mathcal{C}(\mathcal{L})$ satisfies the so called *complete max property*, that (X, τ^{cl}) is discrete if and only if (X, τ^{cl}) is T_1. This result generalizes [2, Theorem 5.34], [2, Theorem 4.28] and [3, Theorem 3.46].

A topological space T is said to be *spectral* [26] iff T is homeomorphic to $Spec(R)$, the prime spectrum of a commutative ring R, with the Zariski topology. Hochster [26] characterized such spaces by giving sufficient and necessarily conditions for a topological space to be spectral. We observe in Proposition 1.49 that if the finer patch topology $(X, \tau^{\mathfrak{fp}})$ is compact, then the classical Zariski topology (X, τ^{cl}) is spectral. Sufficient conditions for $(X, \tau^{\mathfrak{fp}})$ to be compact were provided in Theorems 1.54 and 1.58. Example 1.64 provides several spectra of modules which are shown to be spectral by Theorem 1.54.

In Section 1.5, we restrict our investigations to X-top lattices $\mathcal{L} = (L, \vee, \wedge, 1, 0)$ where $X \subseteq L \backslash \{1\}$. We investigated the interplay between the algebraic properties of $\mathcal{L}$ and the topological space $(X, \tau) = (X, \tau^{cl})$. Several types of compactness and connectedness of (X, τ) are studied in Theorem 1.67. The following short table provides examples of such an interplay.

The results in Chapter One are applied to the complete lattice $LAT(_RM) := (\mathcal{L}(M), \cap, +, 0, M)$ of submodules of a left module M over an associative ring R. In a series of examples 1.73 - 1.78, we apply Theorem 1.70 to a number of spectra $X \subseteq \mathcal{L}(M) \backslash M$ (or $X \subseteq \mathcal{L}(M) \backslash \{0\}$).

Assumption	(X,τ)	$\mathcal{L}$
	all subsets of X are compact	$\mathcal{C}(\mathcal{L})$ satisfies the ACC
	X is clompact	$\mathcal{C}(\mathcal{L})$ satisfies the DCC
	X is T_1	$X = Max(X)$
	$A \subseteq X$ is irreducible	$I(A) \in SI(\mathcal{C}(\mathcal{L}))$
	irreducible	$\sqrt{0} \in SI(\mathcal{C}(\mathcal{L}))$
$SI(\mathcal{C}(\mathcal{L})) \subseteq X$	$V(a)$ is irreducible	$\sqrt{a} \in X$
$SI(\mathcal{C}(\mathcal{L})) \subseteq X$	$V(a)$ is an irreducible component	$\sqrt{a} \in Min(X)$

Table 1: Examples on the Interplay between topological properties of (X,τ) and algebraic properties of $\mathcal{L}$

Let R be a commutative ring and M an R-module. An important tool for studying ${}_RM$ is to decompose it as a (direct) sum of *nice* submodules, if possible. Inspired by the *primary decomposition* of ideals of commutative Noetherian rings [11], many authors studied the primary decompositions of for a proper submodule N of M (assuming that ${}_RM$ is finitely generated) as $N = \bigcap\limits_{i=1}^{n} M_i$ for some primary R-submodules $\{M_1, \cdots, M_n\}$ of M (*e.g.* [13]). Several authors dualized such decompositions and obtained the so called *secondary* (*coprimary*) *representations* of a non-zero module ${}_RM$ as a finite sum $M = \sum\limits_{i=1}^{n} M_i$ of secondary (coprimary) submodules of M; see Macdonald [32] (Kirby [30]).

In Chapter Two we consider *second representable* modules, *i.e.* modules which can be written as finite sums $M = \sum\limits_{i=1}^{n} M_i$ of second submodules $M_1, \cdots, M_n$ of ${}_RM$ (recall that $N \leq M$ is said to be *second* iff $IN = N$ or $IN = 0$ for every $I \leq R$). Sufficient conditions for the existence of second representations for ${}_RM$ are provided in Proposition 2.39 and Theorems 2.44 and 2.49. Examples of modules which are second representable but neither semisimple nor second are provided (*e.g.* 2.47). Several other examples showing that some of the sufficient conditions in the results mentioned above are not necessary (*e.g.* Examples 2.45 and 2.46).

Since every second module is secondary, the First and the Second Uniqueness Theorems (Theorems 2.30 and 2.31, respectively) for a second representable R-module follow from the corresponding ones on secondary representations ([30], [13]). As a consequence of Theorem 2.60, it follows that a second representable Noetherian R-module is a finite direct sum of second submodules. Theorem 2.63 investigates the relation between semisimple, multiplication and second representable modules.

In Chapter Three, we study bounded lattices $\mathcal{L} := (L, \wedge, \vee, 0, 1)$ lattices with an action of a poset $(S, \leq)$. We define several notions of primeness for elements of $L \backslash \{1\}$ (and coprimeness for elements in $L \backslash \{0\}$). In Theorem 3.10, we prove that the spectrum $Spec^c(\mathcal{L})$ of coprime elements in L is nothing but the spectrum $Spec^s(\mathcal{L}^0)$ of second elements in the dual bounded lattice $\mathcal{L}^0 := (L, \vee, \wedge, 1, 0)$. In Section 3.2, we consider the case $\mathcal{L} := LAT(M)$ of submodules of a module M over a commutative ring R. We present the notion of a *pseudo strongly hollow submodule* (*PS-hollow* for short) $N \leq M$ as a weaker notion of that of a *dual primary submodule* of M (i.e. a primary module in $\mathcal{L}^0$). Modules which are finite sums of

PS-submodules are said to be *PS-hollow representable*. Proposition 3.21 asserts the existence of minimal PS-hollow representations for PS-hollow representable modules over Artinian rings. The First and the Second Uniqueness Theorems of minimal pseudo strongly hollow representations are provided in Theorems 3.26 and 3.27, respectively. Sufficient conditions for ${}_RM$ to have a PS-hollow representation are given in Proposition 3.32. Finally, Theorem 3.37 investigates semisimple modules each PS-hollow submodules of which is simple.

CHAPTER 1

ZARISKI-LIKE TOPOLOGIES FOR COMPLETE LATTICES

1.1 Lattices

1.1 *([25]) A* lattice $\mathcal{L}$ *is a* poset $(L, \leq)$ *closed under two binary commutative associative and idempotent operations* $\wedge$ *(*meet*) and* $\vee$ *(*join*), and we write* $\mathcal{L} = (L, \wedge, \vee)$*. We say that a lattice* $(L, \wedge, \vee)$ *a* complete lattice *iff* $\bigwedge_{x \in H} x$ *and* $\bigvee_{x \in H} x$ *exist for any* $H \subseteq L$*. For two lattices* $\mathcal{L} = (L, \wedge, \vee)$ *and* $\mathcal{L}' = (L', \wedge', \vee')$*, a* homomorphism of lattices *from* $\mathcal{L}$ *to* $\mathcal{L}'$ *is a map* $\varphi : L \longrightarrow L'$ *that preserves finite meets and finite joins,* i.e.

$$\varphi(x \wedge y) = \varphi(x) \wedge' \varphi(y) \text{ and } \varphi(x \vee y) = \varphi(x) \vee' \varphi(y) \ \forall x, y \in L.$$

If $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ *and* $\mathcal{L}' = (L', \wedge', \vee', 0', 1')$ *are complete lattices, then a* morphism of complete lattices *from* $\mathcal{L}$ *to* $\mathcal{L}'$ *is a map* $\varphi : L \longrightarrow L'$ *that preserves arbitrary meets and arbitrary joins.*

1.2 *Let* $\mathcal{L} = (L, \wedge, \vee)$ *be a lattice. If* $\mathcal{L}$ *has a* maximum element 1 *and a* minimum element 0, *then* $\mathcal{L}$ *is called a* bounded lattice *and we write* $\mathcal{L} = (L, \wedge, \vee, 0, 1)$. *An element* $x \in L\backslash\{1\}$ *is called* maximal *in* $\mathcal{L}$ *iff* $y = x$ *or* $y = 1$ *whenever* $x \leq y$; *dually, an element* $x \in L\backslash\{0\}$ *is called* minimal *iff* $y = x$ *or* $y = 0$ *whenever* $y \leq x$. *Notice that every complete lattice is bounded. We make the convention that* $\bigwedge_{x\in\emptyset} x = 1$ *and* $\bigvee_{x\in\emptyset} x = 0$.

1.3 *For every lattice* $\mathcal{L} = (L, \wedge, \vee)$, *there is associated the* dual lattice $\mathcal{L}^0 = (L, \wedge^0, \vee^0)$ *where* $\wedge^0 = \vee$ *and* $\vee^0 = \wedge$. *Indeed, if* $\mathcal{L} = (L, \wedge, \vee)$ *is a complete lattice, then the dual lattice* $\mathcal{L}^0$ *is complete. Moreover, if* $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ *is a bounded lattice, then the dual lattice* $\mathcal{L}^0 = (L, \wedge^0, \vee^0, 0^0, 1^0)$ *is bounded with* $0^0 = 1$ *and* $1^0 = 0$.

Example 1.4 *Let* R *be a ring.*

(1) $S = (Ideal(R), \cap, +, R, 0)$, *where* $Ideal(R)$ *is the set of all (two-sided) ideals of* R *is a complete lattice.*

(2) *For any left* R*-module* M, *the set* $LAT(M) = (\mathcal{L}(M), \cap, +, M, 0)$ *is a complete lattice where* $\mathcal{L}(M)$ *is the class of all* R*-submodules of* M.

1.5 *Let* $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ *be a complete lattice.*

(1) *An element* $x \in L\backslash\{1\}$ *is said to be:*

irreducible *[7] iff for any* $a, b \in L$ *with* $a \wedge b = x$, *we have* $a = x$ *or* $b = x$;

strongly irreducible *[7] iff for any* $a, b \in L$ *with* $a \wedge b \leq x$, *we have* $a \leq x$ *or* $b \leq x$.

We denote the set of strongly irreducible elements in L *by* $SI(\mathcal{L})$.

(2) *An element* $x \in L\backslash\{0\}$ *is said to be:*

hollow *iff whenever for any* $a, b \in L$ *with* $x = a \vee b$, *we have* $x = a$ *or* $x = b$;

strongly hollow *[7] iff for any* $a, b \in L$ *with* $x \leq a \vee b$, *we have* $x \leq a$ *or* $x \leq b$.

We denote the set of strongly hollow elements in L *by* $SH(\mathcal{L})$.

(3) *We say that* $\mathcal{L}$ *is*

a hollow lattice *iff* 1 *is hollow (*i.e. *for any two elements* $x, y \in L\backslash\{1\}$ *we have* $x \vee y \neq 1$*);*

a uniform lattice *iff* 0 *is uniform (*i.e. *for any two elements* $x, y \in L\backslash\{0\}$ *we have* $x \wedge y \neq 0$*).*

1.2 X-top Lattices

From now on, we assume that $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a complete lattice.

1.6 *Let* $X \subseteq L\backslash\{1\}$. *For* $a \in L$, *we define the variety of* a *as* $V(a) := \{p \in X \mid a \leq p\}$ *and set* $V(\mathcal{L}) := \{V(a) \mid a \in L\}$. *Indeed,* $V(\mathcal{L})$ *is closed under arbitrary* *intersections (in fact,* $\bigcap_{a\in A} V(a) = V(\bigvee_{a\in A}(a))$ *for any* $A \subseteq L$*). The lattice* $\mathcal{L}$ *is called* X-top *(or a* topological lattice *[1]) iff* $V(\mathcal{L})$ *is closed under finite unions. The lattice* $\mathcal{L}$ *is called* strongly X-top *iff* $X \subseteq SI(\mathcal{L})$ *[1].*

1.7 *Let* $X \subseteq L\backslash\{1\}$*. For any* $Y \subseteq X$*, we set* $I(Y) := \bigwedge\limits_{p\in Y} p$ *and* $\sqrt{a} := I(V(a))$*. We say that* a *is an* X-radical element *iff* $\sqrt{a} = a$*. The set of* X*-radical elements of* L *is*

$$\mathcal{C}^X(L) := \{a \in L \mid \sqrt{a} = a\}.$$

When X *is clear from the context, we drop it from the above notation. Notice that* $\mathcal{C}(\mathcal{L}) = (\mathcal{C}(L), \wedge, \tilde{\vee}, \sqrt{0}, 1)$ *is a complete lattice, where* $\tilde{\bigvee} Y := IV(\bigvee(Y))$ *for any* $Y \subseteq \mathcal{C}(L)$*, i.e.* $\tilde{\bigvee}_{x\in Y} x = \sqrt{\bigvee\limits_{x\in Y} x}$*. It was proved in [1, Theorem 2.2] that* $\mathcal{L}$ *is an* X*-top lattice if and only if the map*

$$V : (\mathcal{C}(L), \wedge, \tilde{\vee}, 1, \sqrt{0}) \longrightarrow (\mathcal{P}(X), \cap, \cup, X, \emptyset),\ \ a \mapsto V(a)$$

is an anti-homomorphism of lattices, that is

$$V(a \wedge b) = V(a) \cup V(b) \text{ and } V(a \vee b) = V(a) \cap V(b) \text{ for all } a, b \in \mathcal{C}(L).$$

The following lemma appeared in [1] except for (2) which is clear.

Lemma 1.8 *Let* $X \subseteq L\backslash\{1\}$*. For any* $x, y \in L$ *and* $A, B \subseteq L$ *we have:*

(1) $A \subseteq B \Rightarrow I(B) \leq I(A)$.

(2) $V(x) \subseteq V(y) \Leftrightarrow \sqrt{y} \leq \sqrt{x}$*. It follows that* $V(x) = V(y) \Leftrightarrow \sqrt{y} = \sqrt{x}$*.*

(3) $V(x) = V(\sqrt{x})$.

(4) $\bigcap_{x \in A} V(x) = V(\bigvee_{x \in A} (x))$.

(5) $I \circ V \circ I = I$.

(6) $V \circ I \circ V = V$.

(7) $\mathcal{L}$ *is* X*-top* $\Longleftrightarrow V(x) \cup V(y) = V(x \wedge y)$ *for any* $x, y \in \mathcal{C}(L)$.

1.9 *Let* $X \subseteq L \backslash \{1\}$ *and set* $\tau := \{X \backslash V(a) \mid a \in L\}$. *We define* τ^{cl} *to be the topology constructed on* X *by taking* τ *as a* subbase, *that is* τ^{cl} *is the set of all arbitrary unions of finite intersections of elements in* τ, *and is called the* classical Zariski topology *on* X. *Moreover,* $\mathcal{L}$ *is* X*-top (*i.e. τ *is closed under finite intersections) if and only if* $\tau^{cl} = \tau$.

1.10 *Let* $Y \subseteq L \backslash \{0\}$. *For any* $a \in L$, *we define the* dual variety $V^0(a) := \{q \in Y \mid q \leq a\}$ *and set* $V^0(\mathcal{L}) = \{V^0(a) \mid a \in L\}$. *We say that* $\mathcal{L}$ *is* dual Y-top *iff the dual lattice* $\mathcal{L}^0$ *is a* Y*-top lattice. For any subset* $A \subseteq Y$, *we set* $H(A) := \bigvee_{q \in A} q$; *also we set* $\sqrt{a}^0 := H(V^0(a))$, *and* $\mathcal{H}(\mathcal{L}) = \mathcal{C}^Y(\mathcal{L}^0)$. *The* dual classical Zariski topology τ^{dcl} *on* Y *is constructed by taking* $\tau^0 := \{Y \backslash V^0(a) \mid a \in L\}$ *as a subbase for this topology. With this process, one can dualize the results obtained in this chapter for the (classical) Zariski-topology to results on the dual (classical) Zariski topology.*

The following lemma recovers [2, 5.14 and 4.10], [3, 3.23] and [4, 3.21].

Lemma 1.11 *Let $X \subseteq L\backslash\{1\}$ and assume that $\mathcal{L}$ is an X-top lattice. The closure of any $Y \subseteq X$ is given by $\overline{Y} = V(I(Y))$.*

Proof. Let $Y \subseteq X$. Notice that $\overline{Y} = V(a)$ for some $a \in L$, whence $a \leq \bigwedge_{p \in Y} p = I(Y)$ and so $V(I(Y)) \subseteq V(a) = \overline{Y}$. On the other hand, $Y \subseteq V(I(Y))$ and so $\overline{Y} \subseteq V(I(Y))$. ∎

1.12 *A non-empty topological space (T, τ) is said to be:*

(1) connected *iff T is* not *the union of two disjoint non-empty open subsets (equivalently, T is* not *the union of two disjoint non-empty closed sets).*

(2) hyperconnected *(or* irreducible *[16]) iff no two non-empty open sets in T are disjoint (equivalently, T is* not *the union of two closed subsets).*

(3) ultraconnected *[16] iff no two non-empty closed sets in T are disjoint.*

1.13 *Let (T, τ) be a topological space. A subset $A \subseteq T$ is called* hyperconnected *[16] (or* irreducible*) iff A is so when considered as a topological space w.r.t. the relative topology induced from (T, τ) (equivalently, A is non-empty and for any two closed subsets F_1, F_2 in T with $A \subseteq F_1 \cup F_2$, we have $A \subseteq F_1$ or $A \subseteq F_2$). The empty set is* not *considered to be irreducible. A closed subset $F \subseteq T$ is said to have a* generic point *$g \in T$ [16] iff $\overline{\{g\}} = F$. The topological space (T, τ) is called* sober *iff every closed irreducible subset of T has a* unique *generic point.*

1.14 *A subset $A \subseteq T$ is irreducible if and only if the closure $\overline{A}$ is irreducible. An* irreducible component *[16] is an irreducible subset of X which is not a proper* *subset of any irreducible subset of T (hence an irreducible component of T is indeed a closed subset).*

The following result generalizes [14, 3.2 and 3.3].

Proposition 1.15 *Let $X \subseteq L\backslash\{1\}$ and consider (X, τ^{cl}).*

(1) *For each $p \in X$, we have $\overline{\{p\}} = V(p)$.*

(2) *$V(p)$ is irreducible $\forall p \in X$.*

(3) *If $Y \subseteq X$ is closed, then $Y = \bigcup\limits_{p \in Y} V(p)$.*

(4) *If $\mathcal{L}$ is X-top, then for any closed subset $Y \subseteq X$ we have $Y = \bigcup\limits_{p\in Y} V(p) = V(\bigwedge\limits_{p \in Y} p)$.*

Proof. Consider (X, τ^{cl}).

(1) Observe that $V(p)$ is closed in (X, τ^{cl}), whence $\overline{\{p\}} \subseteq V(p)$. On the other hand, suppose that $\overline{\{p\}} = \bigcap_{i\in I}(\bigcup_{j=1}^{j=n_i}(V(x_{ij})))$, where $x_{ij} \in L$. Since $p \in \bigcup_{j=1}^{j=n_i}(V(x_{ij}))$ $\forall i \in I$, it follows that $V(p) \subseteq \bigcup_{j=1}^{j=n_i}(V(x_{ij}))$ $\forall i \in I$. Therefore, $V(p) \subseteq \overline{\{p\}}$. Clearly, $\overline{\{p\}} \subseteq V(p)$, whence $\overline{\{p\}} = V(p)$.

(2) Notice that $\{p\}$ is irreducible, whence $V(p) = \overline{\{p\}}$ is irreducible.

(3) Clear.

(4) Let $Y \subseteq X$ be closed. It follows from (3) that $Y = \bigcup\limits_{p\in Y} V(p) \subseteq V(\bigwedge\limits_{p\in Y} p)$. Since $\mathcal{L}$ is assumed to be X-top, $Y = V(x)$ for some $x \in L$ and so $x \leq \bigwedge\limits_{p\in Y} p$, whence $V(\bigwedge_{p\in Y} p) \subseteq V(x) = Y$. Consequently, $Y = \bigcup\limits_{p\in Y} V(p) = V(\bigwedge\limits_{p\in Y} p)$.

Example 1.16 *Consider the complete ideal lattice* $\mathcal{L} = (Ideal(\mathbb{Z}), \cap, +, \mathbb{Z}, 0)$. *Consider* $X = Spec^p(\mathbb{Z})$, *the prime spectrum of* $\mathbb{Z}$. *It is clear that* (X, τ) *is a topological space (the usual Zariski topology on the spectrum of the commutative ring* $\mathbb{Z}$*). Notice that for* $Y := Spec^p(\mathbb{Z})\backslash\{0\}$, *we have* $\overline{Y} = V(I(Y)) = V(\bigcap_{P \in Y} P) = V(0) = X \neq \bigcup_{p \in Y} V(p)$. *This example shows that [14, Proposition 3.1] fails to hold even for domains. However, the proof of Proposition 1.15 provides a correct proof [14, Corollary 2.3] without using [14, Proposition 3.1].*

The following result recovers [14, Proposition 3.8], [3, Proposition 3.24 (1)] and [2, Proposition 5.15 (i)].

Proposition 1.17 *Let* $X \subseteq L\backslash\{1\}$ *and consider* (X, τ^{cl}).

(1) X *is* T_0.

(2) *Every finite closed irreducible subset of* X *has a unique generic point. If* X *is finite, then* X *is sober.*

Proof.

(1) Let $p_1, p_2 \in X$ be such that $\overline{\{p_1\}} = \overline{\{p_2\}}$, whence $V(p_1) = V(p_2)$ and it follows that $p_1 = p_2$, which proves that X is T_0 (notice that, in general, (X, τ) is T_0 if and only if $\overline{\{p_1\}} = \overline{\{p_2\}} \Leftrightarrow p_1 = p_2$).

(2) In general, If (X, τ) is T_0, then every finite irreducible closed subset has a unique generic point. To see this, suppose that F is a closed irreducible finite

set that has no generic point. Pick $x_1 \in F$, whence $\overline{\{x_1\}} \neq F$ and so there is $x_2 \in F\backslash\overline{\{x_1\}}$. Observe that $\overline{\{x_1\}} \cup \overline{\{x_2\}} \neq F$ as F is irreducible. So, there is $x_3 \in F\backslash(\overline{\{x_1\}} \cup \overline{\{x_2\}})$. by continuing in this process, we conclude that F is infinite, a contradiction. The uniqueness of the generic point follows directly from the fact that T_0.

∎

The following observation generalizes [14, Proposition 2.3].

Remark 1.18 *Let $X \subseteq L\backslash\{1\}$. The following are equivalent for (X, τ^{cl}) :*

(1) $\mathcal{L} = \mathcal{C}(\mathcal{L})$.

(2) *For all $x_1, x_2 \in L$ with $V(x_1) = V(x_2)$, we have $x_1 = x_2$.*

Proof. $(1 \Rightarrow 2)$ Suppose $V(x_1) = V(x_2)$ for some $x_1, x_2 \in L$. It follows that $x_1 \leq p, \forall p \in V(x_2)$ whence $x_1 \leq \sqrt{x_2} = x_2$. Similarly, $x_2 \leq x_1$.

$(2 \Rightarrow 1)$ $\forall x \in L$ we have $V(x) = V(\sqrt{x})$, whence $x = \sqrt{x}$. ∎

1.19 *Let $X \subseteq L\backslash\{1\}$ and denote by $Min(X)$ the set of minimal elements of X and by $Max(X)$ the set of maximal elements of X. We say that X is*

atomic *iff for every $p \in X$ there is $q \in Min(X)$ such that $q \leq p$;*

coatomic *iff for every element $p \in X$ there is $q \in Max(X)$ such that $p \leq q$.*

Remarks 1.20 *Let $X \subseteq L\backslash\{1\}$ and consider (X, τ^{cl}).*

(1) *If X satisfies the DCC, then X is atomic.*

(2) *If X is atomic, then there is a subset $A \subseteq X$ such that $X = \bigcup_{p\in A} V(p)$ with $V(p)$ and $V(q)$ are not comparable for any $p \neq q \in A$ (e.g. take $A = Min(X)$).*

(3) *Let X be atomic and $Min(X)$ finite. Then X is irreducible if and only if $Min(X)$ is a singleton. To prove this, observe that $X = \bigcup\limits_{p\in Min(X)} V(p)$ with $p \nleq q$ for any $p \neq q$ are in $Min(X)$. Clearly, X is irreducible if and only if $Min(X)$ is a singleton.*

Remarks 1.21 *Let $X \subseteq L\backslash\{1\}$ with $0 \in X$ and consider (X, τ^{cl}).*

(1) *If $F \subseteq X$ is closed and $0 \in F$, then $F = X$. To prove this, observe that $X = V(0) = \overline{\{0\}} \subseteq F$.*

(2) *Every non-empty open subset of X contains 0. To see this, let $O \subseteq X$ be open. If $0 \notin O$, then $0 \in F := X\backslash O$. By (1), $X\backslash O = X$, i.e. $O = \emptyset$.*

(3) *X is irreducible since $Min(X) = \{0\}$, a singleton (see Remark 1.20 (3)).*

It was proved in [1, Proposition 2.7], that if $\mathcal{L}$ is an X-top lattice and $A \subseteq X$ is such that $I(A)$ is irreducible in $(\mathcal{C}(L), \wedge)$, then A is irreducible in (X, τ). The following result proves a stronger version of the converse.

Proposition 1.22 *Let $X \subseteq L\backslash\{1\}$ and assume that $\mathcal{L}$ is an X-top lattice. If $A \subseteq X$ is irreducible, then $I(A)$ is strongly irreducible in $(\mathcal{C}(L), \wedge)$.*

Proof. Suppose that $a \wedge b \leq I(A)$ for some $a, b \in \mathcal{C}(L)$. Now, $\overline{A} = V(I(A)) \subseteq V(a \wedge b) \overset{[1,\text{ Theorem } 2.2]}{=} V(a) \cup V(b)$. Since A is irreducible, $\overline{A}$ is also irreducible, whence $\overline{A} \subseteq V(a)$ or $\overline{A} \subseteq V(b)$. So, $a = I(V(a)) \leq I(\overline{A}) = I(V(I(A))) = I(A)$ or $b = I(V(b)) \leq I(\overline{A}) = I(V(I(A))) = I(A)$. ∎

Corollary 1.23 *Let $X \subseteq L\backslash\{1\}$ and assume that $\mathcal{L}$ be an X-top lattice. The following conditions are equivalent for $A \subseteq X$:*

(1) *A is irreducible;*

(2) *$I(A)$ is strongly irreducible in $(\mathcal{C}(L), \wedge)$;*

(3) *$I(A)$ is irreducible in $(\mathcal{C}(L), \wedge)$.*

1.24 *A* maximal element *in $\mathcal{L}$ is a maximal element in the poset $(L\backslash\{1\}, \leq)$. An element $x \in L$ is called* minimal *in $\mathcal{L}$ iff x is maximal in $\mathcal{L}^0$. We denote by $Max(\mathcal{L})$ (resp. $Min(\mathcal{L})$) the set of all maximal (resp. minimal) elements in $\mathcal{L}$. The lattice $\mathcal{L}$ is called* coatomic *iff for every element $x \in L\backslash\{1\}$, there exists $y \in Max(\mathcal{L})$ such that $x \leq y$. Dually, $\mathcal{L}$ is called* atomic *iff for every element $x \in L\backslash\{0\}$, there exists $y \in Min(\mathcal{L})$ such that $y \leq x$.*

Let $A \subseteq L$. The lattice $\mathcal{L}$ is said to have the complete A-property *iff $\bigwedge_{p \in A\backslash\{q\}} p \nleq q$ for any $q \in A$. The lattice $\mathcal{L}$ is said to have the* complete max property *iff L has the complete $Max(\mathcal{L})$-property.*

Lemma 1.25 *Let $\mathcal{L}$ be an X-top lattice. If $\mathcal{L}$ is coatomic and $Max(\mathcal{L}) \subseteq X$, then $Max(\mathcal{L}) = Max(X)$.*

Proof. Let $p \in Max(X)$. Since $\mathcal{L}$ is coatomic, there is $y \in Max(\mathcal{L})$ such that $p \leq y$ and so $p = y$ as $Max(\mathcal{L}) \subseteq X$. ∎

The following result recovers and generalizes [3, Proposition 3.45], [2, Propositions 5.33, 4.27], and [4, Proposition 3.40]. The additional conditions assumed in these results imply that $Max(\mathcal{L}) = Max(X)$ (or $Min(\mathcal{L}) = Min(X)$ in the dual cases).

Proposition 1.26 *Let $X \subseteq L\backslash\{1\}$. The following are equivalent for (X, τ^{cl}) :*

(1) *X is T_1;*

(2) *$Max(X) = X = Min(X)$.*

Proof. X is T_1 $\Leftrightarrow$ every singleton is closed $\Leftrightarrow$ $\{p\} = \overline{\{p\}} = V(p)$ $\forall p \in X$ $\Leftrightarrow$ $Max(X) = X = Min(X)$. ∎

Theorem 1.27 *Let $X \subseteq L\backslash\{1\}$ and consider (X, τ^{cl}). Then $Max(X) = Max(\mathcal{C}(\mathcal{L}))$. Moreover, the following conditions are equivalent:*

(1) *X is T_1 and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property;*

(2) *X is discrete.*

Proof. Let $p \in Max(X)$. Then $p \in \mathcal{C}(L)$ and so $p \in Max(\mathcal{C}(\mathcal{L}))$; otherwise, there is $x \in \mathcal{C}(L)\backslash\{1\}$ such that $p \lneq x$. Since $x \neq 1$, there is $q \in X$ such that $x \leq q$ and so $p \lneq q$ (a contradiction). For the reverse inclusion, let $x \in Max(\mathcal{C}(L))$. Notice that $x = \bigwedge_{p \in A} p$ for some $\emptyset \neq A \subseteq X$. Since $A \subseteq \mathcal{C}(L)$, it follows by the maximality of x in $\mathcal{C}(L)$ that $x = \bigwedge_{p \in A} p = q$ for some $q \in A$, i.e. A is singleton and $x \in X$. Moreover, $x \in Max(X)$ as $X \subseteq \mathcal{C}(L)$.

(1) $\Rightarrow$ (2) : Assume that $\mathcal{C}(\mathcal{L})$ satisfies the complete max property. Since $Max(X) = Max(\mathcal{C}(L))$, we have $\bigwedge_{p \in Max(X)\setminus\{q\}} p \nleq q$ for any $q \in Max(X)$. Notice that for any $q \in X$, we have $X = V(\bigwedge_{p \in X\setminus\{q\}} p) \cup \{q\}$ and by our assumption $q \notin V(\bigwedge_{p \in X\setminus\{q\}} p)$. Hence, every singleton in X is open, that is (X, τ^{cl}) is discrete.

(2) $\Rightarrow$ (1) : Assume that X is discrete and show that $\mathcal{C}(\mathcal{L})$ satisfies the complete max property. To show this, suppose that $q \in X$ and let $Y = X\setminus\{q\}$. Observe that

$$Y = \overline{Y} = V(I(Y))$$

as $\{q\}$ is open. Hence, $I(Y) \nleq q$, which completes the proof as $X = Max(X) = Max(\mathcal{C}(\mathcal{L}))$. ∎

The following result generalizes [2, Theorem 5.34], [2, Theorem 4.28] and [3, Theorem 3.46].

Corollary 1.28 *Let $X \subseteq L\setminus\{1\}$ and consider (X, τ^{cl}). If $\mathcal{C}(\mathcal{L})$ satisfies the complete max property, then the following conditions are equivalent:*

(1) $Max(X) = X = Min(X)$;

(2) *X is T_2;*

(3) *X is T_1;*

(4) *X is discrete.*

Corollary 1.29 *Let $X \subseteq L\backslash\{1\}$ and consider (X, τ^{cl}). Assume that $\mathcal{L}$ satisfies the complete max property, $\mathcal{L}$ is coatomic and $Max(\mathcal{L}) \subseteq \mathcal{C}(L)$. The following are equivalent:*

(1) *$Max(X) = X = Min(X)$.*

(2) *X is T_2.*

(3) *X is T_1.*

(4) *X is discrete.*

Proof. Let $p \in Max(\mathcal{C}(\mathcal{L}))$. Since $\mathcal{L}$ is coatomic, there exists $q \in Max(\mathcal{L})$ such that $p \leq q$. By assumption, $Max(\mathcal{L}) \subseteq \mathcal{C}(L)$ whence $p = q$. So, $Max(\mathcal{L}) = Max(\mathcal{C}(\mathcal{L}))$ and the results follows by Theorem 1.28. ∎

A topological space is *regular* [37] iff any non-empty closed set F and any point x that does not belong to F can be separated by disjoint open neighborhoods. A T_3 space is one which is both T_1 and regular. In general, regular spaces need not be Hausdorff. However, we have a special situation.

Remark 1.30 *If (X, τ^{cl}) is regular, then (X, τ^{cl}) is T_3. To see this, assume that X is regular and let $p \neq q$ be elements in X. Assume, without loss of generality, that $p \nleq q$ so that is, $q \notin V(p)$. Since X is regular, there are two disjoint open sets O_1 and O_2 in X such that $q \in O_1$ and $V(p) \subseteq O_2$. Therefore X is T_2.*

A topological space X is *normal* [37] iff any two disjoint closed sets of X can be separated by disjoint open neighborhoods. The following example shows that the normality of (X, τ^{cl}) does not guarantee that it is regular.

Example 1.31 *Let R be a local ring with $|Spec(R)| \geq 2$. Then $Spec(R)$ is normal because it has no disjoint non-empty closed sets. However, $Spec(R)$ is not T_1 whence not regular by Remark 1.30. To see this, notice that the assumption $|Spec(R)| \geq 2$ implies that there is a prime ideal p of R and a maximal ideal q of R such that $p \lneqq q$. Hence, every open set containing q contains p as well.*

1.3 Examples

Throughout this section, R is an associative ring, M is a non-zero left R-module and $LAT(M) := (\mathcal{L}(M), \cap, +, M, 0)$ the complete lattice of R-submodules of M. Moreover, we denote by $Max(M)$ (resp. $\mathcal{S}(M)$) the possibly empty set of maximal (resp. simple) R-submodules of M. By $L \leq M$, we mean that L is an R-submodule of M. With abuse of notation, we mean by $I \leq R$ that I is a (two sided) ideal of R.

1.32 *Let M be a left R-module. We call an R-submodule $K \leq M$:*

prime *[20] iff $K \neq M$ and for any $N \leq M$ and $I \leq R$, we have*

$$IN \subseteq K \Rightarrow N \subseteq K \text{ or } IM \subseteq K. \tag{1.1}$$

first *[1] iff $K \neq 0$ and for any $N \leq K$ and $I \leq R$, we have*

$$IN = 0 \Rightarrow N = 0 \text{ or } IK = 0.$$

coprime *[2] iff* $K \neq M$ *and for any* $I \leq R$, *we have*

$$IM + K = M \text{ or } IM \subseteq K.$$

second *[2] iff* $K \neq 0$ *and for any* $I \leq R$ *we have*

$$IK = K \text{ or } IK = 0.$$

By $Spec^{p}(M)$ *(resp.* $Spec^{f}(M)$, $Spec^{c}(M)$, $Spec^{s}(M)$*) we denote the spectrum of prime (resp. first, coprime, second) R-submodules of* M.

1.33 *An R-submodule* K *of* M *is said to be* fully invariant *[3] (and we write* $L \leq^{f.i.} M$*) iff* $f(L) \subseteq L$ *for all* $f \in S := End(M)$. *If every submodule of* M *is fully invariant, then* M *is said to be a* duo module *[3]. For any* $L_1, L_2 \leq M$, *we define*

$$L_1 * L_2 = \sum_{f \in Hom(M, L_2)} f(L_1) \quad \text{and} \quad L_1 \odot_M L_2 = \bigcap_{f \in S, f(L_1) = 0} f^{-1}(L_2);$$

see [3] and [4]. Notice that if $L_1 \leq^{f.i.} M$, *then* $L_1 * L_2 \subseteq L_1 \cap L_2$.

1.34 *A fully invariant submodule* $K \leq^{f.i.} M$ *is:*

fully prime *in* M *[3] iff* $K \neq M$ *and for any* $L_1, L_2 \leq^{f.i.} M$, *we have*

$$L_1 * L_2 \subseteq K \Rightarrow L_1 \subseteq K \text{ or } L_2 \subseteq K.$$

fully coprime *in* M *[4] iff* $K \neq 0$ *and for any* $L_1, L_2 \leq_R^{f.i.} M$ *we have*

$$K \subseteq L_1 \odot_M L_2 \Rightarrow K \subseteq L_1 \text{ or } K \subseteq L_2.$$

By $Spec^{fp}(M)$ *(resp.* $Spec^{fc}(M)$*) we denote the spectrum of fully prime (resp. fully coprime) R-submodules of* M*.*

The following example summarizes some facts about some Zariski-like topologies on several spectra of submodules of a given module.

Example 1.35 *Consider* $X_1 := Spec^p(M)$, $X_2 := Spec^c(M)$, $X_3 := Spec^{fp}(M)$, $X_4 := Spec^s(M)$, $X_5 := Spec^f(M)$ *and* $X_6 := Spec^{fc}(M)$*. Notice that* $X_1, X_2, X_3 \subseteq \mathcal{L}(M) \backslash \{M\}$ *and so one can construct the classical Zariski topology* τ_-^{cl} *on any of them as we did for general complete lattices* $\mathcal{L} = (L, \wedge, \vee, 1, 0)$ *and* $X \subseteq L \backslash \{1\}$*. On the other hand, one can construct dual classical Zariski topologies on* τ_-^{dcl} *only any of* $X_4, X_5, X_6 \subseteq \mathcal{L}(M) \backslash \{0\}$*. Moreover,* M *is* top^p-module *(resp. a* top^c-module, *a* top^{fp}-module*) if and only if* $LAT(M)$ *is* X_1*-top (resp.* X_2*-top,* X_3*-top). On the other hand,* M *is a* top^s-module *(resp. a* top^f-module, *a* top^{fc}*-module) iff* $LAT(M)$ *is dual* X_4*-top (resp. dual* X_5*-top, dual* X_6*-top). The following table summarize some facts about these spaces.*

Type	$M \notin Spec^{-}(M)$	$0 \notin Spec^{-}(M)$
Subsets of L	$X_1 = Spec^{p}(M)$, $X_2 = Spec^{c}(M)$, $X_3 = Spec^{fp}(M)$	$X_4 = Spec^{f}(M)$, $X_5 = Spec^{s}(M)$, $X_6 = Spec^{fc}(M)$
Variety $V^{-}(N)$	$\{P \in Spec^{-}(M) \mid N \leq P\}$	$\{P \in Spec^{-}(M) \mid P \leq N\}$
Subbase τ_{-}	$\{X \backslash V^{-}(N) \mid N \leq M\}$	$\{X \backslash V^{-}(N) \mid N \leq M\}$
τ^{cl} or τ^{dcl}	τ_{-}^{cl}: the topology generated by τ_{-}	τ_{-}^{dc}: the topology generated by τ_{-}
(Dual) X-top	$\mathcal{L}$ is X_i-top $\Leftrightarrow \tau_{-} = \tau_{-}^{cl}$	$\mathcal{L}^0$ is X_j-top $\Leftrightarrow \tau_{-} = \tau_{-}^{dcl}$

Table 1.1: Examples of spectra of submodules of a given module

Example 1.36 *Let M be a local left module over an arbitrary ring R, i.e. M has a unique maximum proper submodule (e.g. the $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, p is any prime and k is any positive integer). Consider $X_1 = Spec^{p}(M)$ and $X_2 = Spec^{c}(M)$. Then $\mathcal{C}^{X_1}(LAT(M))$ and $\mathcal{C}^{X_2}(LAT(M))$ satisfy the complete max property (notice that any maximal submodule is prime and coprime).*

Example 1.37 *Consider $\mathcal{L} := LAT(M)$, $X_1 = Spec^{p}(M)$ and $X_2 = Spec^{c}(M)$. Every maximal submodule of M is a prime and a coprime submodule, i.e. $Max(M) \subseteq X_1$ and $Max(M) \subseteq X_2$. So, it is enough to assume that ${}_RM$ is a coatomic module satisfying the complete max property to satisfy the equivalent conditions of Corollary 1.29. Moreover, Theorem 1.28 applies if $RAD^{p}(M) := \mathcal{C}^{X_1}(\mathcal{L})$ (resp. $RAD^{c}(M) := \mathcal{C}^{X_2}(\mathcal{L})$) satisfies the complete max property as a lattice.*

Example 1.38 *Consider $\mathcal{L} := LAT(M)$, $X_4 = Spec^{f}(M)$ and $X_5 = Spec^{s}(M)$. Every simple submodule of M is a second and a first submodule of M, i.e. $\mathcal{S}(M) \subseteq$*

X_4 and $\mathcal{S}(M) \subseteq X_5$. So, it is enough to assume that ${}_RM$ is an atomic module with the complete min property to satisfy the equivalent conditions of Corollary 1.29 applied to $\mathcal{L}^0$. Moreover, Theorem 1.28 applies if $\mathcal{C}^{X_4}(\mathcal{L}^0)$ (resp. $\mathcal{C}^{X_5}(\mathcal{L}^0)$) satisfies the complete min property as a lattice.

Remark 1.39 *It was proved in [2], that if ${}_RM$ is a coatomic top^c-module satisfying the complete max property, then*

$$Spec^c(M) = Max(M) \Longleftrightarrow X \text{ is } T_2 \Leftrightarrow X \text{ is } T_1 \Leftrightarrow X \text{ is discrete.}$$

A similar result was proved for $Spec^{fp}(M)$ assuming that ${}_RM$ is a self projective coatomic duo module $(S-PCD)$. Notice that it was proved in [3, Remark 3.12] that if ${}_RM$ is self projective and duo, then every maximal submodule is fully prime. Other conditions were assumed on M in the dual cases to ensure that $\mathcal{S}(M) = Min(X)$. So, Theorem 1.28 generalizes all the corresponding results in [3] and [2].

1.4 Spectral Spaces

As before, $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a complete lattice.

1.40 *A topological space is said to be* spectral *[26] iff it is homeomorphic to Spec(R), the prime spectrum of a commutative ring R with the Zariski topology. Hochster [26, Proposition 4] characterized such spaces. A topological space (X, τ) is spectral if and only if all of the following four conditions are satisfied:*

(1) *X is compact;*

(2) *X has a basis of compact open sets closed under finite intersections and*

(3) *X is sober.*

Remark 1.41 *Let $X \subseteq L \backslash \{1\}$. If X is finite, then (X, τ^{cl}) is spectral: By Proposition 1.17, X is T_0 and sober. The remaining Hochster's conditions 1.40 follow directly from the finiteness of X.*

Definition 1.42 *Let $X \subseteq L \backslash \{1\}$ and consider (X, τ^{cl}). Set*

$$\mathcal{R}(\mathcal{L}) := \{\sqrt{x} \mid x \in L \text{ and } V(x) \text{ is irreducible}\}. \tag{1.2}$$

We say that X satisfies the radical condition *iff $\mathcal{R}(\mathcal{L}) \subseteq X$.*

Lemma 1.43 *Let $X \subseteq L \backslash \{1\}$ and consider (X, τ^{cl}). If X is sober, then X satisfies the radical condition. The converse holds if $\mathcal{L}$ is X-top.*

Proof. Let X be sober. Let $x \in \mathcal{R}(\mathcal{L})$. Since X is sober, $V(x) = \overline{\{p\}} \overset{1.15}{=} V(p)$ for some $p \in X$. It follows by Lemma 1.8 (2) that $\sqrt{x} = p \in X$.

For the converse, assume that $\mathcal{L}$ is X-top. Let F be a closed irreducible subset of X. Since $\mathcal{L}$ is X-top, $F = V(x)$ for some $x \in L$. By our hypothesis, $\sqrt{x} \in X$. By Lemma 1.8 (3), $F = V(x) = V(\sqrt{x})$ and so $\sqrt{x}$ is the unique generic point of F (the uniqueness is obvious). Therefore, X is sober. ∎

Proposition 1.44 *Let $X \subseteq L \backslash \{1\}$ and assume that $\mathcal{L}$ is an X-top lattice. If $\mathcal{L}$ satisfies the ACC, then every subset of (X, τ) is compact.*

Proof. Let $A \subseteq X$ and suppose that $\mathcal{O} = \{X\backslash V(x_i)) \mid x_i \in L,\, i \in I\}$ is an open cover for A. Since $\mathcal{L}$ satisfies the ACC, $\bigvee_{i\in I} x_i = \bigvee_{j\in J} x_j$ for some finite subset $J \subseteq I$. Notice that

$$A \subseteq \bigcup_{i\in I}(X\backslash V(x_i)) = X\backslash V(\bigvee_{i\in I} x_i)) = X\backslash V(\bigvee_{i\in I} x_i)) = \bigcup_{i\in J}(X\backslash V(x_j)),$$

i.e. $\{X\backslash V(x_j) \mid j \in J\}$ is finite subcover of $\mathcal{O}$ for A. ∎

Proposition 1.45 *Let $X \subseteq L\backslash\{1\}$ and assume that $\mathcal{L}$ is an X-top lattice. The following conditions are equivalent:*

(1) *$\mathcal{C}(\mathcal{L})$ satisfies the ACC;*

(2) *Every subset of (X, τ) is compact;*

(3) *Every open set in (X, τ) is compact.*

Proof. $(1 \Rightarrow 2)$: Consider the complete lattice $(\mathcal{C}(L), \wedge, \tilde{\vee}, \sqrt{0}, 1)$. Since $V(x) = V(\sqrt{x})$ for every $x \in L$, we conclude that $\mathcal{C} := (\mathcal{C}(L), \wedge, \tilde{\vee}, \sqrt{0}, 1)$ is an X-top lattice. By our assumption, $\mathcal{C}$ satisfies the ACC and so every subset of X is compact by Proposition 1.44.

$(3 \Rightarrow 1)$: Let $a_1 \leq a_2 \leq a_3 \leq \cdots$ be an ascending chain in $\mathcal{C}(\mathcal{L})$. Notice that $X\backslash V(a_1) \subseteq X\backslash V(a_2) \subseteq X\backslash V(a_3) \subseteq \cdots$. Setting $b = \tilde{\bigvee}_{i=1}^{\infty} a_i$, we observe that

$$X\backslash V(b) = X\backslash V(\bigvee_{i=1}^{\infty} a_i) = X\backslash \bigcap_{i=1}^{\infty} V(a_i) = \bigcup_{i=1}^{\infty}(X\backslash V(a_i)).$$

By our assumption, the open set $X\backslash V(b)$ is compact and so $X\backslash V(b) = \bigcup_{i=1}^{n} X\backslash V(a_i) = X\backslash V(a_n)$ for some $n \in \mathbb{N}$, i.e. $b = a_n$ and the ascending chain stabilizes. ∎

Corollary 1.46 *Let $X \subseteq L\backslash\{1\}$ and $\mathcal{L}$ be an X-top lattice such that $\mathcal{C}(\mathcal{L})$ satisfies the ACC. Then (X, τ) is spectral $\Leftrightarrow$ (X, τ) is sober.*

Proof. By Proposition 1.17 X is T_0. The result follows now using Proposition 1.45 and Hochster's characterization for spectral spaces 1.40. ∎

In [15], the so called *finer patch topology* was used to prove that for any left module M over an associative ring R, and $X = Spec^{p}(M)$, the classical Zariski topology (X, τ^{cl}) is a spectral space provided that ACC holds for intersections of prime submodules of M.

1.47 *Let $X \subseteq L\backslash\{1\}$. The finer patch topology $\tau^{\mathfrak{fp}}$ on X is the one whose subbase is*

$$\mathcal{B} = \{V(x) \cap X\backslash V(y) \mid x, y \in L\}. \tag{1.3}$$

It is clear that $\tau^{cl} \subseteq \tau^{\mathfrak{fp}}$. So, if $\tau^{\mathfrak{fp}}$ is compact, then τ^{cl} is compact.

Example 1.48 *Let $\mathcal{P}$ be the set of all prime numbers in $\mathbb{Z}$ and consider the ring $R = \prod_{p\in\mathcal{P}} \mathbb{Z}_p$. Then the finer patch topology associated with $Spec(R)$ is not compact while, trivially, the classical Zariski topology is compact. In general, if R is a ring with zero dimension and $Spec(R)$ is infinite, then the finer patch topology associated with $Spec(R)$ is not compact while, trivially, the classical Zariski topology is compact.*

Proposition 1.49 *Let $X \subseteq L\backslash\{1\}$. If $(X, \tau^{\mathfrak{fp}})$ is compact, then (X, τ^{cl}) is spectral.*

Proof. Assume that $(X, \tau^{\mathfrak{fp}})$ is compact. We apply Hochster's characterizations of spectral spaces to prove that (X, τ^{cl}) is spectral. Notice that (X, τ^{cl}) is T_0 by Proposition 1.17 and is compact since $\tau^{cl} \subseteq \tau^{\mathfrak{fp}}$.

Claim I: (X, τ^{cl}) is sober.

Let $Y \subseteq X$ be a closed irreducible set in (X, τ^{cl}). Then $Y \overset{1.15}{=} \bigcup\limits_{p \in Y} V(p)$. On the other hand Y is closed in $(X, \tau^{\mathfrak{fp}})$, whence compact in $(X, \tau^{\mathfrak{fp}})$ (recall that every closed subset of a compact space is compact). Therefore, the open cover $\mathcal{O} := \{V(p) : p \in Y\}$ of Y has a finite subcover $\{V(p_1), \cdots, V(p_n)\}$, i.e. $Y = \bigcup\limits_{i=1}^{i=n} p_i$. But Y is irreducible, whence $Y = V(p_k)$ for some $k \in \{1, 2, \cdots, n\}$. Clearly, p_k is the unique generic point of Y.

Claim II: X has a basis of compact open sets closed under finite intersections.

We prove this claim in two steps.

Step 1: Every basic open subset of (X, τ^{cl}) is compact.

Let B be a basic open subset of (X, τ^{cl}), i.e. $B = \bigcap\limits_{i=1}^{i=n} X\backslash V(x_i)$ for some $\{x_1, \cdots, x_n\} \subseteq L$. Observe that $X\backslash V(x_i)$ is closed in $(X, \tau^{\mathfrak{fp}})$ $\forall i \in \{1, 2, \cdots, n\}$. So, B is closed in (X, τ), whence compact in $(X, \tau^{\mathfrak{fp}})$. Since $\tau^{cl} \subseteq \tau^{\mathfrak{fp}}$, B is compact in (X, τ^{cl}).

Step 2: The collection of open compact subsets of (X, τ^{cl}) is closed under arbitrary intersections.

Let U be an open compact set in (X, τ^{cl}). Then we can write $U =$

$\bigcup_{i=1}^{n} \bigcap_{j=1}^{m_i} X \backslash V(x_{ij})$ for some subset $\{x_{ij} \mid j = 1, 2, \cdots, m_i,\ i = 1, \cdots, n\}$ (the union is finite because of the compactness of U). Notice that U is closed in $(X, \tau^{\mathfrak{fp}})$. So, any intersection of open compact subsets in (X, τ^{cl}) is closed in $(X, \tau^{\mathfrak{fp}})$; so it is compact in $(X, \tau^{\mathfrak{fp}})$, whence compact in (X, τ^{cl}). ∎

Example 1.50 *The ring of integers $\mathbb{Z}$ is Noetherian and so the finer patch topology on Spec($\mathbb{Z}$) is compact because the ACC is satisfied on the radical ideals by [15, Theorem 2.2]. This example shows that $(X, \tau^{\mathfrak{fp}})$ can be compact although X is infinite.*

Example 1.51 *Let L be infinite and be such that the elements of $X := L\backslash\{0, 1\}$ are not comparable (notice that for all $a \neq b$ in X we have $a \wedge b = 0$ and $a \vee b = 1$). Notice that $(X, \tau^{\mathfrak{fp}})$ is not compact, whereas (X, τ^{cl}) is compact because it is the cofinite topology on X. Notice also that $\mathcal{C}(\mathcal{L})$ satisfies the ACC and every element in $\mathcal{C}(\mathcal{L})$ can be written as an irredundant meet of elements in X, but this guarantees the compactness for the finer patch topology. Observe that $\mathcal{L}$ is not X-top and (X, τ^{cl}) is not sober and hence not spectral.*

Proposition 1.52 *Let $X \subseteq L\backslash\{1\}$ and consider (X, τ^{cl}). If $V(x)$ is reducible for some $x \in L$, then $V(x) = \bigcup_{i=1}^{n} V(x_i)$ for some elements $x_1, x_2, \cdots, x_n \in L$, where $V(x_i) \subsetneqq V(x)$ for all $i = 1, 2, \cdots, n$.*

Proof. Let $V(x)$ be reducible for some $x \in L$, i.e. $V(x) = F_1 \cup F_2$ where both F_1 and F_2 are closed proper subsets of $V(x)$. Suppose that $F_1 = \bigcap_{i \in I} \bigcup_{j=1}^{n_i} V(x_{ij})$ and $F_2 = \bigcap_{l \in L} \bigcup_{k=1}^{m_l} V(y_{lk})$ for some $\{x_{ij}\}, \{y_{lk}\} \subseteq L$. Since F_1 and F_2 are proper subsets of $V(x)$, we have $V(x) \nsubseteq \bigcup_{j=1}^{n_{i_0}} V(x_{i_0 j})$ for some $i_0 \in I$ and $V(x) \nsubseteq \bigcup_{k=1}^{m_{l_0}} V(y_{l_0 k})$ for some $l_0 \in L$, whence $V(x) \neq \bigcup_{j=1}^{n_{i_0}} V(x_{i_0 j}) \cap V(x)$ and $V(x) \neq \bigcup_{k=1}^{m_{l_0}} V(y_{l_0 k}) \cap V(x)$. Set $x_r := x_{i_0 r} \vee x$ for $r = 1, 2, \cdots, n_{i_0}$ and $x_{n_{i_0}+r} = y_{l_0 r} \vee x$ for $r = 1, 2, \cdots, m_{l_0}$ and let $n := n_{i_0} + m_{l_0}$. By construction, $V(x) = \bigcup_{r=1}^{n} V(x_r)$, where each $V(x_r)$ is a proper subset of $V(x)$. ∎

As a direct consequence of Proposition 1.52, we obtain the following result which recovers [14, Proposition 2.26] proved for the prime spectrum of a module over a ring.

Corollary 1.53 *Let $X \subseteq L \backslash \{1\}$ and assume that $|X| \geq 2$. If (X, τ^{cl}) is T_2, then there are $x_1, x_2, \cdots, x_n \in L$ such that $X = \bigcup\limits_{i=1}^{n} V(x_i)$, while $X \neq V(x_i)$ for all $i = 1, 2, \cdots, n$.*

The radical condition is automatically satisfied by the spectrum of prime submodules of a given left module over an associative ring by [14, Theorem 3.4, Corollary 3.6]. However, we need to check it when dealing with other cases.

The following result generalizes [15, Theorem 3.2]:

Theorem 1.54 *Let $X \subseteq L \backslash \{1\}$ and consider (X, τ^{cl}). If X satisfies the radical condition, then $\mathcal{C}(\mathcal{L})$ satisfies the ACC if and only if $(X, \tau^{\mathfrak{fp}})$ is compact. It follows that, If $\mathcal{C}(\mathcal{L})$ satisfies the ACC and X satisfies the radical condition, then (X, τ^{cl}) is spectral.*

Proof. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the ACC and that X satisfies the radical condition. We need only to prove that $(X, \tau^{\mathfrak{fp}})$ is compact since it follows then by Proposition 1.49 that (X, τ^{cl}) is spectral.

Suppose that $(X, \tau^{\mathfrak{fp}})$ is not compact, i.e. there is an open cover $\mathcal{A}$ in $\tau^{\mathfrak{fp}}$ for X which does not have a finite subcover for X.

Let

$$\mathbb{E} := \{x \in \mathcal{C}(L) \mid V(x) \text{ is not covered by a finite subcover of } \mathcal{A}\}.$$

Observe that $\sqrt{0} \in \mathbb{E}$, i.e. $\mathbb{E} \neq \emptyset$. Since $\mathcal{C}(\mathcal{L})$ satisfies the ACC, $\mathbb{E}$ has a maximal element p. Notice that $V(p) \neq \emptyset$.

Case 1: $p \notin X$. Since X satisfies the radical condition, $V(p)$ is reducible and it follows by Proposition 1.52 that $V(p) = \bigcup_{i=1}^{n} V(x_i)$ for some $x_1, \cdots, x_n \in \mathcal{C}(L)$ (see Lemma 1.8 (3)), where $V(x_i) \subsetneqq V(p)$, whence $p \lneqq x_i$, for all $i \in \{1, \cdots, n\}$. Since p is maximal in $\mathbb{E}$ and $p \lneqq x_i$, $V(x_i)$ is covered by a finite subcover of $\mathcal{A}$ for all $i \in \{1, \cdots, n\}$. Hence $V(p)$ is covered by a finite subcover of $\mathcal{A}$, a contradiction.

Case 2: $p \in X$. It follows that $p \in O$ for some $O \in \mathcal{A}$ and so $p \in B$, where B is a basic open subset of O. Assume

$$B = \bigcap_{i=1}^{n} (V(x_i) \cap X \backslash V(y_i)) \text{ for some subset } \{x_1, \cdots, x_n, y_1, \cdots, x_n\} \subseteq L.$$

Observe that $z_i := y_i \vee p \nleq p$ as $y_i \nleq p \; \forall \; i \in \{1, 2, \cdots n\}$.

Claim: $V(p) \cap \bigcap_{i=1}^{n} X \backslash V(z_i) \subseteq B$. To prove this claim, let $q \in V(p) \cap \bigcap_{i=1}^{n} X \backslash V(z_i)$ for all $i \in \{1, 2, \cdots, n\}$, whence $p \leq q$ and $y_i \vee p \nleq q$ for all $i \in \{1, 2, \cdots, n\}$. It follows that $p \leq q$ and $y_i \nleq q$ for all $i \in \{1, 2, \cdots, n\}$. But $x_i \leq p$ for all $i \in \{1, 2, \cdots, n\}$ whence $x_i \leq q$ and $y_i \nleq q$ for all $i \in \{1, 2, \cdots, n\}$, i.e. $q \in \bigcap_{i=1}^{n}(V(x_i) \cap X\backslash V(y_i)) = B$ as claimed.

Now, notice that for all $i \in \{1, 2, \cdots, n\}$, we have $p \lneq z_i$ and so $V(z_i)$ is covered by a finite subcover $\mathcal{A}_i$ of $\mathcal{A}$. On the other hand, $V(p)\backslash \bigcup_{i=1}^{n} V(z_i) = V(p) \cap \bigcap_{i=1}^{n} X\backslash V(z_i) \subseteq B \subseteq O$. Hence $\{O\} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_n$ is a finite subcover of $\mathcal{A}$ for $V(p)$, which is a contradiction.

Therefore, $(X, \tau^{\mathfrak{fp}})$ is compact.

Conversely, assume that $(X, \tau^{\mathfrak{fp}})$ is compact. Suppose that $\mathcal{C}(\mathcal{L})$ does not satisfy the ACC. Then there is an infinite strictly increasing chain $a_1 \lneq a_2 \lneq ....$ of elements in $\mathcal{C}(\mathcal{L})$. Since $(X, \tau^{\mathfrak{fp}})$ is compact, $V(a_1)$ is compact as it is closed. But one can check that $\{V(a_i) \cap (X\backslash V(a_{i+1})) \mid i = 1, 2, ...\} \cup \{V(\bigvee_{i=1}^{\infty} a_i)\}$ is an open cover for $V(x_1)$ which does not have a finite subcover, a contradiction. ▌

Remark 1.55 *Let $X \subseteq L\backslash\{1\}$. The radical condition in Theorem 1.54 is necessary for (X, τ^{cl}) to be spectral. Recall that this condition is satisfied if X is sober (see Lemma 1.43).*

Definition 1.56 *Let $X \subseteq L\backslash\{1\}$. An element $p \in X$ is called* minimal in X over *$x \in L$ iff $p = q$ whenever $x \leq q \leq p$ for some $q \in X$.*

Corollary 1.57 *Let $X \subseteq L\backslash\{1\}$. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the ACC, and that for any $x \in \mathcal{C}(L)\backslash(X \cup \{1\})$ with $V(x) \neq \emptyset$ there is a completely strongly irreducible minimal element in X over it with respect to $(\mathcal{C}(L), \wedge)$. Then $(X, \tau^{\mathfrak{fp}})$ is compact (and consequently (X, τ^{cl}) is spectral).*

Proof. We claim that X satisfies the radical condition. Let $x \in \mathcal{R}(\mathcal{L})\backslash X$. In particular, $V(x) \neq \emptyset$. Let p be a completely strongly irreducible minimal element in X over x. Then $\bigwedge\limits_{q\in V(x)\backslash\{p\}} q \nleq p$ (otherwise, $\bigwedge_{q\in V(x)\backslash\{p\}} q \leq p$ and the complete strong irreducibility of p would imply that $q \lneq p$ for some $q \in V(x)$ contradicting the minimality of p over x). Therefore, $V(x) = V(\bigwedge_{q\in V(x)\backslash\{p\}} q) \cup V(p)$ a union of proper closed subsets and so $V(x)$ is reducible, a contradiction. So, X satisfies the radical condition. Now, the hypotheses of Theorem 1.54 are satisfied and it follows that $(X, \tau^{\mathfrak{fp}})$ is compact and consequently (X, τ^{cl}) is spectral. ∎

Theorem 1.58 *Let $X \subseteq L\backslash\{1\}$ and consider (X, τ^{cl}). Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC and that $Min(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$. Then $(X, \tau^{\mathfrak{fp}})$ is compact if and only if $V(p)$ is finite $\forall p \in Min(X)$.*

Proof. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC and that $Min(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$. We show that $(X, \tau^{\mathfrak{fp}})$ is compact. Notice first of all that $X = \bigcup\limits_{p\in Min(X)} V(p)$, since $\mathcal{C}(\mathcal{L})$ satisfies the DCC.

Claim: $Min(X)$ is finite. To prove this claim, notice that $\bigwedge\limits_{p\in Min(X)} p = \bigwedge\limits_{i=1}^{n} p_i$ for some $\{p_1, p_2, \cdots, p_n\} \subseteq Min(X)$ (since $\mathcal{C}(\mathcal{L})$ satisfies the DCC). So, $\bigwedge_{i=1}^{n} p_i \leq p$ for all $p \in Min(X)$. By assumption, $Min(X) \subseteq SI(\mathcal{C}(\mathcal{L}))$, whence $p = p_i$ for some $i \in \{1, 2, \cdots, n\}$. Consequently, $Min(X)$ is finite.

If $V(p)$ is finite $\forall p \in Min(X)$, then X is finite, whence $(X, \tau^{\mathfrak{fp}})$ is compact.

Conversely, suppose that $(X, \tau^{\mathfrak{fp}})$ is compact and that $V(p)$ is infinite for some $p \in Min(X)$.

Case 1: $V(p)$ contains an infinite chain $p = x_1 \lneq x_2 \lneq ....$ which does not stabilize. Consider the open cover $\mathcal{A} := \{V(x_i) \cap (X\backslash V(x_{i+1}) \mid i = 1, 2, ..\} \cup \{V(\bigvee\limits_{i=1}^{\infty} x_i)\}$ for $V(p)$. Clearly $\mathcal{A}$ has no finite subcover for $V(x_1)$, whence $(X, \tau^{\mathfrak{fp}})$

is not compact, a contradiction.

Case 2: $V(p)$ does not contain any infinite chain. It follows that there is an infinite subset $A \subseteq V(p)$ of incomparable elements. Since $\mathcal{C}(\mathcal{L})$ satisfied the DCC, it follows that $\bigwedge_{x \in A} x = \bigwedge_{x \in F} x$ for some finite subset $F \subseteq A$. Since A is infinite, there is $q \in A \backslash F$ such that $p \lneq q$ for some $p \in F$, a contradiction. ∎

Lemma 1.59 *Let $X \subseteq L \backslash \{1\}$ and $\mathcal{L}$ be an X-top lattice. Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC. Then $X \subseteq SI(\mathcal{C}(\mathcal{L}))$.*

Proof. Since $\mathcal{L}$ is an X-top lattice, we have $\tau = \tau^{cl}$. Notice that for every $p \in X$, the singleton $\{p\}$ is irreducible in (X, τ), whence $p = I(\{p\})$ is strongly irreducible in $(\mathcal{C}(L), \wedge)$ by Proposition 1.22. ∎

Corollary 1.60 *Let $X \subseteq L \backslash \{1\}$ and $\mathcal{L}$ be an X-top lattice. If $\mathcal{C}(\mathcal{L})$ satisfies the DCC, then $(X, \tau^{\mathfrak{fp}})$ is compact if and only if $V(p)$ is finite $\forall p \in Min(X)$.*

Proof. Follows directly by applying Lemma 1.59 and Theorem 1.58. ∎

Example 1.61 *Let $\mathcal{L} = (L, \wedge, \vee, 1, 0)$ be a complete lattice, where L is an infinite ascending chain $x_1 \leq x_2 \leq \cdots$ endowed with a maximum element 1 such that $\bigvee_{i \in I} x_i = 1$ for every infinite subset $I \subseteq \mathbb{N}$. Let $X = L \backslash \{1\}$. Then $\mathcal{C}(\mathcal{L})$ satisfies the DCC, and $Min(X) \subseteq SI(\mathcal{C}(\mathcal{L})$. Hence, $\tau^{\mathfrak{fp}}$ is not compact by Theorem 1.58 because $V(x_1)$ is infinite. Moreover, every descending chain of (X, τ) is a spectral subspace.*

In what follows, let R be a ring, M a left R-module and consider $\mathcal{L} := LAT(M)$, the complete lattice of left R-submodules of M.

Example 1.62 *Let $X = Spec^p(M)$, the spectrum of prime R-submodules of M. By [14, Theorem 3.4 (i)], $Spec^p(M)$ satisfies the radical condition. Therefore, Theorem 1.54 recovers [15, Theorem 3.2] as a special case.*

Example 1.63 *Let ${}_RM$ be Noetherian and $X = SI(M)$, the spectrum of strongly irreducible R-submodules of M, whence $\mathcal{L}$ is X-top. By [1, Proposition 2.7], $SI(M)$ satisfies the radical condition. Therefore $(SI(M), \tau^{\mathfrak{fp}})$ is compact and $(SI(M), \tau)$ is spectral.*

Example 1.64 *Applying Theorem 1.54, we obtain several examples of spectral spaces:*

(1) *If ${}_RM$ is duo and $\mathcal{C}(\mathcal{L})$ satisfies the ACC, then $Spec^{fp}(M)$ is spectral (notice that $Spec^{fp}(M)$ satisfies the radical condition by [3, Proposition 3.30]).*

(2) *If ${}_RM$ is duo and $\mathcal{H}(L)$ satisfies the DCC, then $X = Spec^{fc}(M)$ is spectral (notice that $X = Spec^{fc}(M)$ satisfies the radical condition by [4, Proposition 3.25]).*

(3) *If ${}_RM$ is a completely distributive top^c-module and $\mathcal{C}(\mathcal{L})$ satisfies the ACC, then $Spec^c(M)$ is spectral (notice that $X = Spec^c(M)$ satisfies the radical condition by [2, Proposition 5.19 (i)]).*

(4) *If ${}_RM$ is a top^s-module and $\mathcal{H}(L)$ satisfies the DCC, then $Spec^s(M)$ is spectral (notice that $X = Spec^s(M)$ satisfies the radical condition by [2, Proposition 4.14 (i)]).*

(5) *If $_RM$ is a top^f-module, $I(A)$ is first for every irreducible subset $A \subseteq Spec^f(M)$ and $\mathcal{H}(L)$ satisfies the DCC, then $Spec^f(M)$ is spectral (notice that the assumption on the irreducible subsets of $X = Spec^f(M)$ is equivalent to X satisfying the radical condition by [1, Remark 4.25]).*

1.5 Algebraic versus Topological Properties

As before, $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a complete lattice. In this section we study the interplay between the topological properties of (X, τ^{cl}) where $X \subseteq L\backslash\{1\}$ (or (X, τ^{dcl}) where $X \subseteq L\backslash\{0\}$) and the algebraic properties of $\mathcal{L}$. Applications will be given to the special case $\mathcal{L} = LAT({}_RM)$, where R is a ring and M is a left R-module.

1.65 *We say that an element $x \in L$ is is* finitely constructed *in $\mathcal{L}$ iff x cannot be written as an infinite irredundant join of elements of L, that is, for any collection $\{x_i\}_{i\in I} \subseteq L$ such that $\bigvee\limits_{i\in I} x_i = x$, there is a finite sub-collection $\{x_j\}_{j\in J}$ of $\{x_i\}_{i\in I}$ with $\bigvee\limits_{j\in J} x_j = x$. An element x is called* countably finitely constructed *in $\mathcal{L}$ iff x cannot be written as an infinite countable irredundant join of elements of L, i.e. for any countable collection $\{x_i\}_{i\in I} \subseteq L$ with $\bigvee\limits_{i\in I} x_i = x$, there is a finite sub-collection $\{x_j\}_{j\in J}$ of $\{x_i\}_{i\in I}$ with $\bigvee\limits_{j\in J} x_j = x$. An element x is called* countably constructed *in $\mathcal{L}$ iff x cannot be written as an uncountable irredundant join of elements of L.*

In the following result, we collect some remarks.

Remarks 1.66 *Let $\mathcal{L}$ be an X-top lattice, $X \subseteq L\backslash\{1\}$ and consider the topological space (X, τ).*

(1) *The following are equivalent:*

(a) (X, τ) is irreducible;

(b) $\sqrt{0} \in SI(\mathcal{C}(\mathcal{L}))$;

(c) If $X = \bigcup_{i\in I} V(x_i)$, *then either* I *is infinite or there is* $i_0 \in I$ *such that* x_{i_0} *is a lower bound for* X.

(2) (X, τ) *is* T_1 *if and only if* $Max(X) = X$.

(3) (X, τ) *is Noetherian* $\Leftrightarrow$ $\mathcal{C}(\mathcal{L})$ *satisfies the ACC* $\Leftrightarrow$ *each set in* X *is compact* $\Leftrightarrow$ *each open set in* X *is compact.*

(4) (X, τ) *is Artinian* $\Leftrightarrow$ $\mathcal{C}(\mathcal{L})$ *satisfies the DCC* $\Leftrightarrow$ *every closed cover for any subset of* X *has a finite subcover.*

(5) (X, τ) *is (countably) compact if and only if* 1 *is (countably) finitely constructed in* $\mathcal{C}(\mathcal{L})$.

(6) *If* $SI(\mathcal{C}(\mathcal{L})) \subseteq X$, *then* (X, τ) *is sober.*

(7) *If* X *satisfies the radical condition, then* (X, τ) *is sober.*

(8) *Assume that* $\mathcal{C}(\mathcal{L})$ *satisfies the complete max property. Then,* (X, τ) *is* T_1 $\Leftrightarrow$ (X, τ) *is discrete.*

(9) *If* (X, τ) *is an atomic, Lindelof (compact) and* $V(p)$ *is open* $\forall\, p \in Min(X)$, *then* $Min(X)$ *is countable (finite).*

(10) $V(x)$ *is irreducible for every* $x \in X$.

Proof. Let $\mathcal{L}$ be an X-top lattice.

(1) $(a \Leftrightarrow b)$ Apply Corollary 1.23 to $V(0) = X$.

$(a \Rightarrow c)$ Suppose that $X = \bigcup_{i \in I} V(x_i)$ with I finite. Since X is irreducible, $V(x_{i_0}) = X$ for some $i_0 \in I$ whence x_{i_0} a lower bound for X.

$(c \Rightarrow a)$ Suppose that $X = V(x) \cup V(y)$ for some $x, y \in L$. By our assumption, x is a lower bound for X whence $X = V(x)$ or y is a lower X whence $X = V(y)$. Therefore, X is irreducible.

(2) Apply Proposition 1.26 to $(X, \tau) = (X, \tau^{cl})$.

(3) It is easy to check that the first two statements are equivalent. The remaining equivalences follow by applying Proposition 1.15 to $(X, \tau) = (X, \tau^{cl})$.

(4) Notice that any open set in X has the form $X \backslash V(x)$ where $x \in \mathcal{C}(L)$. The equivalence of the first two statements is straightforward. We claim that they are equivalent to the third statement.

Assume that $\mathcal{C}(\mathcal{L})$ satisfies the DCC. Let $U \subseteq X$ and $\{V(x) \mid x \in A\}$ be a closed cover, i.e. $U \subseteq Y := \bigcup_{x \in A} V(x)$, and assume without loss of generality that $A \subseteq \mathcal{C}(L)$. It follows that $I(Y) = \bigwedge_{x \in A} x$. Since $\mathcal{C}(\mathcal{L})$ satisfies the DCC, $I(Y) = \bigwedge_{x \in B} x$ for some finite subset $B \subseteq I$. It follows that

$$\overline{Y} \overset{\text{Lemma 1.11}}{=} V(I(Y)) = V(\bigwedge_{x \in B} x) \overset{\text{[1, Theorem 2.2]}}{=} \bigcup_{x \in B} V(x).$$

Therefore, $U \subseteq \bigcup_{x \in B} V(x)$ for some finite subset $B \subseteq A$.

Conversely, suppose that $x_1 \geq x_2 \geq \cdots$ is a descending chain in $\mathcal{C}(\mathcal{L})$ and consider the induced ascending chain $V(x_1) \subseteq V(x_2) \subseteq \cdots$. Let $Y = \bigcup_{i=1}^{\infty} V(x_i)$. By assumption, $Y = \bigcup_{i=1}^{n} V(x_i)$ for some $n \in \mathbb{N}$, whence $V(x_n) = V(x_m)$ for all $m \geq n$ and consequently $x_n = x_m$ for all $m \geq n$ by Lemma 1.8.

(5) Assume that X is (countably) compact and suppose that $1 = \tilde{\bigvee}_{i \in I} x_i$ where $x_i \in \mathcal{C}(L)$ (and I is countable). It follows that $\emptyset = V(\tilde{\bigvee}_{i \in I} x_i) = \bigcap_{i \in I} V(x_i)$, i.e. $X = \bigcup_{i \in I}(X \backslash V(x_i))$. Since X is (countably) compact, $X = \bigcup_{j \in F}(X \backslash V(x_j))$ for some finite subset F of I and so $1 = \tilde{\bigvee}_{j \in F} x_j$. So, 1 is (countably) finitely constructed. The converse can be obtained similarly.

(6) Let $F \subseteq X$ be a closed irreducible subset. Then $F = V(x)$ for some $x \in L$, whence $\sqrt{x} \in SI(\mathcal{C}(\mathcal{L})) \subseteq X$ by Proposition 1.22. The uniqueness of the generic point is obvious.

(7) This follows by Lemma 1.43.

(8) This follows by applying Theorem 1.28 to $(X, \tau^{cl}) = (X, \tau)$.

(9) Assume that X is Lindelof (compact). Since X is atomic, $X = \bigcup_{p \in Min(X)} V(p)$, whence the open cover $\{V(p) \mid p \in Min(X)\}$ has a countable (finite) subcover for X, i.e. $X = \bigcup_{p \in A} V(p)$ for some countable (finite) subset $A \subseteq Min(X)$. **Claim:** $Min(X) = A$. Let $q \in Min(X)$. Since $X = \bigcup_{p \in A}(V(p)$, we have $q \in V(p)$ for some $p \in A$, whence $q = p$ by the minimality of q. Consequently $Min(X)$ is countable (finite).

(10) This is obtained by applying Proposition 1.15 to $(X, \tau^{cl}) = (X, \tau)$.

Theorem 1.67 *Let $X \subseteq L\backslash\{1\}$ and assume that $\mathcal{L}$ is an X-top lattice.*

(1) *The following are equivalent for the sublattice*

$$\mathcal{C}'(\mathcal{L}) = \{x \in \mathcal{C}(L) \mid x\tilde{\vee}y = 1 \text{ and } x \wedge y = \sqrt{0} \text{ for some } y \in \mathcal{C}(L)\}$$

of $\mathcal{C}(\mathcal{L})$:

(a) (X, τ) *is connected.*

(b) If $x \in L$ is such that $\emptyset \neq V(x) \subsetneq X$, then $V(x)$ is not open.

(c) $V(x) \cap V(y) \neq \emptyset$ for all $x \in L$ such that $\sqrt{x} \notin \{\sqrt{0}, 1\}$ and for all $y \in L$ such that $X\backslash V(x) \subseteq V(y)$.

(d) $\mathcal{C}'(\mathcal{L}) = \{\sqrt{0}, 1\}$.

(2) *Let (X, τ) be T_1. Then X is singleton if and only if (X, τ) is connected and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property.*

(3) *If X is coatomic and $Max(X)$ is countable (finite), then (X, τ) is Lindelof (compact).*

(4) *Let X be coatomic. Then $Max(X)$ is singleton if and only if (X, τ) is connected and each element in $Max(X)$ is completely strongly irreducible in $(\mathcal{C}(L), \wedge)$.*

(5) *Let $\mathcal{L}$ be coatomic and $Max(L) \subseteq X$. Then (X, τ) is ultraconnected if and only if $\mathcal{L}$ is hollow.*

(6) *Let* $\emptyset \neq X$ *be atomic. Then* (X, τ) *is reducible if and only if* $Min(X) = I_1 \cup I_2$ *such that* $\bigwedge_{p \in I_2} p \nleq q_l$ *for some* $q_l \in I_l$ *and* $\bigwedge_{p \in I_1} p \nleq q_2$ *for some* $q_2 \in I_2$.

(7) *Let* $\emptyset \neq X$ *be atomic. Then* (X, τ) *is connected if and only if for every* $\emptyset \neq \mathbf{m} \subsetneq Min(X)$ *there exists some* $q \in X$ *such that*

$$\wedge_{p \in \mathbf{m}} p \bigvee \wedge_{p \in Max(X) \backslash \mathbf{m}} p \leq q.$$

Proof. Let $X \subseteq L \backslash \{1\}$ and assume that $\mathcal{L}$ is an X-top lattice.

(1) Let $x, y \in \mathcal{C}'$. Then there are $x', y' \in \mathcal{C}(L)$ such that $x \tilde{\vee} x' = 1$, $x \wedge x' = \sqrt{0}$, $y \tilde{\vee} y' = 1$ and $y \wedge y' = \sqrt{0}$. One can check that $x \wedge y$ and $x \tilde{\vee} y$ are also in $\mathcal{C}'$ with the corresponding elements $x' \tilde{\vee} y'$ and $x' \wedge y'$ respectively (recall that if $\mathcal{L}$ is X-top then $\mathcal{C}(\mathcal{L})$ is distributive by [1, Theorem 2.2]).

The equivalence $(a) \Leftrightarrow (b)$ is trivial.

$(a \Rightarrow c)$ Let $x, y \in L$ be such that $\sqrt{x} \notin \{\sqrt{0}, 1\}$ and $X \backslash V(x) \subseteq V(y)$. It follows that $V(x) \cup V(y) = X$, whence $V(x) \cap V(y) \neq \emptyset$ (otherwise, X will be disconnected).

$(c \Rightarrow b)$ Suppose that $\emptyset \neq V(x) \subsetneq X$ is open for some $x \in L$, so that $\sqrt{x} \notin \{\sqrt{0}, 1\}$. Let $y \in L$ be such that $X \backslash V(x) = V(y)$. By our assumption, $V(x) \cap V(y) \neq \emptyset$ (a contradiction).

$(c \Rightarrow d)$ Let $x \in \mathcal{C}'(\mathcal{L})$. Then there is $y \in \mathcal{C}(L)$ such that $x \wedge y = \sqrt{0}$ and $x \tilde{\vee} y = 1$. Clearly, x and y satisfy the conditions stated in (c), whence

$V(x\tilde{\vee}y) = V(x) \cap V(y) \neq \emptyset$, i.e. $x\tilde{\vee}y \neq 1$, which is a contradiction.

$(d \Rightarrow a)$ Suppose that $V(x) \cup V(y) = X$, $V(x) \cap V(y) = \emptyset$ for some $x, y \in L$, and assume without loss of generality that $x, y \in \mathcal{C}(L)$. It is easy to show that $x, y \in \mathcal{C}'(\mathcal{L})$, and it follows by (d) that $V(x) = X$ or $V(x) = \emptyset$.

(2) Let (X, τ) be T_1. If $\mathcal{C}(\mathcal{L})$ satisfies the complete max property, then applying 1.28 to $(X, \tau) = (X, \tau^{cl})$, we conclude that X is discrete. If X is moreover connected, then X is indeed a singleton. The converse is trivial.

(3) Let X be coatomic and $Max(X)$ be countable (finite). Let $\mathcal{A} = \{X \backslash V(x) \mid x \in A\}$ be an open cover for X. Then $\bigcap\limits_{x \in A} V(x) = \emptyset$ and so for any $p \in Max(X)$, there exists $x_p \in A$ such that $p \notin V(x_p)$. **Claim:** $\bigcap\limits_{p \in Max(X)} V(x_p) = \emptyset$. Suppose that $q \in \bigcap\limits_{p \in Max(X)} V(x_p)$. Since X is coatomic, $q \leq \tilde{p}$ for some $\tilde{p} \in Max(X)$ and so $\tilde{p} \in \bigcap\limits_{p \in Max(X)} V(x_p)$, a contradiction.

It follows that $X = \bigcup\limits_{p \in Max(X)} (X \backslash V(x_p))$, i.e. $\{X \backslash V(x_p \mid p \in Max(X)\}$ is a countable (finite) subcover of $\mathcal{A}$ for X.

(4) Let X be coatomic.

$(\Rightarrow)$ Assume that $Max(X) = \{p\}$. For all $q \in X$, $q \leq p$ as X is coatomic and so p is completely irreducible in the $(\mathcal{C}, \wedge)$. Also, if $X = V(x) \cup V(y)$ and $V(x), V(y) \neq \emptyset$, then $p \in V(x) \cap V(y)$ and so X is connected.

$(\Leftarrow)$ Suppose that $|Max(X)| \geq 2$ and let $Max(X) = \mathbf{M}' \cup \mathbf{M}''$ with $\mathbf{M}' \cap$

$\mathbf{M}'' = \emptyset$ for some $\emptyset \neq \mathbf{M}' \subsetneq Max(X)$. Set

$$A := \{p \in X \mid p \leq q \text{ for some } q \in \mathbf{M}' \text{ and } p \nleq q \; \forall \; q \in \mathbf{M}''\},$$

$B := X \backslash A$, $x := \bigwedge_{p \in A} p$ and $y := \bigwedge_{p \in B} p$.

Claim: $V(x) \cap V(y) = \emptyset$.

Suppose that $\tilde{p} \in V(x) \cap V(y)$, whence $y \leq \tilde{p} \leq \tilde{q}$ for some $\tilde{q} \in Max(X)$. Since $\tilde{q}$ is completely strongly irreducible, $\tilde{q} \in \mathbf{M}'$: otherwise, $\tilde{q} \in \mathbf{M}''$ and $x = \bigwedge_{p \in A} p \leq \tilde{q}$ implies that $p' \leq \tilde{q} \in \mathbf{M}''$ for some $p' \in A$, a contradiction. Hence, $y \leq \tilde{q} \in \mathbf{M}'$. Similarly, since $\tilde{q}$ is completely strongly irreducible, $q' \leq \tilde{q}$ for some $q' \in B$, which is a contradiction. Therefore $V(x) \cap V(y) = \emptyset$, and $V(x)$ and $V(y)$ are non-empty ($\mathbf{M}' \subseteq V(x)$ and $\mathbf{M}'' \subseteq V(y)$) with $V(x) \cup V(y) = X$, whence X is disconnected.

(5) Let $\mathcal{L}$ be coatomic and $Max(L) \subseteq X$.

($\Rightarrow$) Assume that X is ultraconnected. Let $x, y \in L \backslash \{1\}$. Since $\mathcal{L}$ is coatomic, there are $p, q \in Max(\mathcal{L}) \subseteq X$ with $x \leq p$ and $y \leq q$, whence $V(x)$ and $V(y)$ are non-empty. By assumption, X is ultraconnected, whence $V(x \vee y) = V(x) \cap V(y) \neq \emptyset$. Hence $x \vee y \neq 1$. Consequently, $\mathcal{L}$ is hollow.

($\Leftarrow$) Assume that $\mathcal{L}$ is hollow. Let $V(x)$ and $V(y)$ be non-empty closed subsets for some $x, y \in L$. Then $x, y \in L \backslash \{1\}$, whence $x \vee y \neq 1$ as $\mathcal{L}$ is hollow. Since L is coatomic, $x \vee y \leq q$ for some $q \in Max(\mathcal{L}) \subseteq X$. Hence $V(x) \cap V(y) = V(x \vee y) \neq \emptyset$. Therefore, X is ultraconnected.

(6) Let X be reducible, i.e. $X = V(x) \cup V(y)$ for some $x, y \in L$ such that $V(x) \subsetneq X$ and $V(y) \subsetneq X$. Set

$$I_1 = \{p \in Min(X) \mid x \leq p\} \text{ and } I_2 = \{p \in Min(X) \mid y \leq p\}.$$

Since X is atomic, $\sqrt{x} = \bigwedge_{p \in I_1} p$ and $\sqrt{y} = \bigwedge_{p \in I_2} p$. Indeed, $\sqrt{x} \not\leq q_2$ for some $q_2 \in I_2$: otherwise, $\sqrt{x} \leq p \,\forall\, p \in I_2$ and it follows that $V(x) = X$. Similarly, $\sqrt{y} \not\leq q_1$ for some $q_1 \in I_1$. The converse is trivial.

(7) Let $\emptyset \neq X$ be atomic.

($\Rightarrow$) Assume that X is connected. Let $\emptyset \neq \mathbf{m} \subsetneq Min(X)$, $x := \wedge_{p \in \mathbf{m}} p$ and $y = \wedge_{p \in Max(X) \backslash \mathbf{m}} p$. Since X is atomic, $X = V(x) \cup V(y)$. Since X is connected, $V(x \vee y) = V(x) \cap V(y) \neq \emptyset$, i.e. $\exists\, q \in X$ such that $x \vee y \leq q$.

($\Leftarrow$) Suppose that $X = V(x) \cup V(y)$ for some $x, y \in L$. Set

$$\mathbf{m}' := \{p \in Min(X) \cap V(x)\} \text{ and } \mathbf{m}'' := Min(X) \backslash \mathbf{m}'.$$

Case 1: $\mathbf{m}' = \emptyset$. In this case, $X = V(y)$.

Case 2: $\mathbf{m}' = Min(X)$. In this case, $X = V(x)$.

Case 3: $\emptyset \neq \mathbf{m}' \subsetneq Min(X)$. By our assumption, $\sqrt{x} \vee \sqrt{y} \leq q$ for some $q \in X$ and so

$$V(x) \cap V(y) = V(\sqrt{x}) \cap V(\sqrt{y}) = V(\sqrt{x} \vee \sqrt{y}) \neq \emptyset.$$

Consequently, X is connected.

$\blacksquare$

Example 1.68 *Let M be a left module over an arbitrary ring R. Consider $X_1 = Spec^p(M)$ and $X_2 = Spec^c(M)$. Suppose that $\sqrt{0} = 0$ (e.g. the $\mathbb{Z}$-module $\mathbb{Z}[x]$). Then the set $\mathcal{C}'$ which was described in Theorem 1.67 (1) is the set of the prime radical direct summands (resp. the coprime radical direct summands).*

Corollary 1.69 *Let $X \subseteq L\backslash\{1\}$ and assume that $\mathcal{L}$ is an X-top lattice.*

(1) *Let X be atomic, coatomic with $|Max(X)| \leq |Min(X)|$ and $V(p)$ is open $\forall\ p \in Min(X)$, then (X, τ) is Lindelof (compact) if and only if $Max(X)$ is countable (finite).*

(2) *Let $X = Max(\mathcal{L})$. Then $|Max(\mathcal{L})| = 1$ if and only if (X, τ) is connected and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property.*

Proof.

(1) If (X, τ) is Lindelof, then $Min(X)$ is countable by Remark 1.66(10). Conversely, assume that $Max(X)$ is countable (finite). Let $\mathcal{O} = \{X\backslash V(x) \mid x \in A \subseteq L\}$ be an open cover for X, i.e. $\bigcap\limits_{x \in A} V(x) = \emptyset$ and assume without loss of generality that $V(x) \neq \emptyset$ for each $x \in A$ (If $V(y) = \emptyset$ for some $y \in A$, then $\{X\backslash V(y)\}$ is a finite subcover of X). Pick $x' \in A$ and set $\mathbf{M} := \{q \in Max(X) \mid x' \leq q\}$. Observe that $\mathbf{M}$ is non-empty as $V(x') \neq \emptyset$ and X is coatomic. For each $q \in \mathbf{M}$, pick $X\backslash V(x_q) \in \mathcal{O}$ that contains q.

Claim: $x' \vee \bigvee_{q \in \mathbf{M}} x_q \nleq p$ for each $p \in Max(X)$.

Case (1): $p \in \mathbf{M}$. In this case, $x_p \nleq p$ and so $x' \vee \bigvee_{q \in \mathbf{M}} x_q \nleq p$.

Case (2): $p \in Max(X) \backslash \mathbf{M}$. In this case, $x' \nleq p$ and so $x' \vee \bigvee_{q \in \mathbf{M}} x_q \nleq p$.

Therefore, $V(x' \vee \bigvee_{q \in \mathbf{M}} x_q) = \emptyset$ and

$$\{X \backslash V(x')\} \cup \{X \backslash V(x_q) \mid q \in \mathbf{M}\}$$

is a countable (finite) subcover of $\mathcal{O}$ as $Max(X)$ is countable (finite).

(2) Assume that $Max(\mathcal{L}) = X$, whence $Max(X) = X = Max(\mathcal{L})$. It follows by Theorem 1.28 that X is T_1. So, we can use now Theorem 1.67 (2).

∎

Theorem 1.70 *Let $X \subseteq L \backslash \{1\}$ and assume the $\mathcal{L}$ is an X-top lattice.*

(1) *There is a 1-1 correspondence*

$$\mathcal{C}(L) \quad \longleftrightarrow \quad \textit{closed sets in } (X, \tau).$$

(2) *If $SI(\mathcal{C}(\mathcal{L})) \subseteq X$, then there is a 1-1 correspondence*

$$X \quad \longleftrightarrow \quad \textit{Irreducible closed sets in } (X, \tau).$$

(3) *If* $SI(\mathcal{C}(\mathcal{L})) \subseteq X$*, then there is a 1-1 correspondence*

$$Min(X) \quad \longleftrightarrow \quad \textit{Irreducible components in } (X, \tau).$$

Proof. Since $\mathcal{L}$ is X-top, the set of closed sets in X is given by $\mathcal{V} = \{V(y) \mid y \in L\}$. Define

$$\begin{aligned} f &: \mathcal{C}(L) \longrightarrow \mathcal{V},\ x \mapsto V(x); \\ g &: \mathcal{V} \longrightarrow \mathcal{C}(L),\ V(y) \mapsto \sqrt{y}. \end{aligned}$$

(1) For any $x \in \mathcal{C}(L)$ and $y \in L$, we have

$$\begin{aligned} (g \circ f)(x) &= g(V(x)) = \sqrt{x} = x; \\ (f \circ g)(V(y)) &= f(\sqrt{y}) = V(\sqrt{y}) = V(y). \end{aligned}$$

So, f provides a 1-1 correspondence $\mathcal{C}(L) \longleftrightarrow \mathcal{V}$ with inverse g.

(2) Consider the restrictions of f to X and of g to the class of irreducible closed varieties.

For every $x \in X$, the variety $V(x)$ is irreducible by Proposition 1.15 (2). On the other hand, if $V(y)$ is irreducible for some $y \in L$, then $\sqrt{y}$ is strongly irreducible in $\mathcal{C}(\mathcal{L})$ by Proposition 1.22, whence $\sqrt{y} \in X$ by our assumption.

(3) Consider the restrictions of f to $Min(X)$ and of g to the class of irreducible

components in (X,τ).

For every $x \in Min(X)$. By (2), $V(x)$ is irreducible. Suppose that $V(x) \subseteq V(y)$ for some $y \in L$ with $V(y)$ irreducible. Since $SI(\mathcal{C}(\mathcal{L})) \subseteq X$, it follows by (2) that $\sqrt{y} \in X$, whence $\sqrt{y} \leq x$. However, x in minimal in X, whence $x = \sqrt{y}$ and $V(x) = V(y)$.

On the other hand, let A be an irreducible component in (X,τ). Any irreducible component is closed. Moreover, $I(A)$ is strongly irreducible in $\mathcal{C}(\mathcal{L})$ as A is irreducible, hence $I(A) \in X$. Suppose that $p \leq I(A)$ for some $p \in X$. It follows that $A = \overline{A} = V(I(A)) \subseteq V(p) = \overline{\{p\}}$. However, $V(p)$ is irreducible as it is the closure of a singleton, so $V(p) = A$ as A is an irreducible component. So, $p = I(A)$. Consequently, $I(A) \in Min(X)$.

$\blacksquare$

Example 1.71 *The first correspondence ($\mathcal{C}(\mathcal{L}(M)) \longleftrightarrow$ closed sets in (X,τ)) of Theorem 1.70 holds for any $X \subseteq \mathcal{L}(M)\backslash\{M\}$ such that $\mathcal{L}$ is X-top, as well as for any $X \subseteq \mathcal{L}(M)\backslash\{0\}$ such that $\mathcal{L}$ is dual X-top. So, this result recovers [2, 4.12 and 5.16], [3, 3.27] and [4, 3.23] as special cases.*

The following table summarizes some of the results we obtained in this section. Some of them generalize results in the literature on Zariski-like topologies for left modules over associative rings, which can be recovered now as special cases. At several occasions, our results were obtained under conditions and assumptions weaker than those in the corresponding results in the literature.

Assumption & location	X**-top lattice** $\mathcal{L}$	(X, τ)	**Results recovered**
Proposition 1.26	$Max(X) = X$	T_1	[2, 4.27, 5.33], [3, 3.45])
Proposition 1.45	$\mathcal{C}(\mathcal{L})$ satisfies the ACC	Each set in X is compact	
Remark 1.66 (3)	$\mathcal{C}(\mathcal{L})$ satisfies the ACC	Noetherian	[2, 4.12, 5.16])
Proposition 1.45	$\mathcal{C}(\mathcal{L})$ satisfies the ACC	Each open set in X is compact	
Remark 1.66 (4)	$\mathcal{C}(\mathcal{L})$ satisfies the DCC	Artinian	[2, 4.12, 5.16]
Remark 1.66 (4)	$\mathcal{C}(\mathcal{L})$ satisfies the DCC	Every closed cover for any subset of X has a finite subcover	
Theorem 1.27	$Max(X) = X$ and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property	Discrete	[2, 4.28, 5.34], [3, 3.46], [1, 4.33])
Theorem 1.67 (1)	$\mathcal{C}' = \{\sqrt{0}, 1\}$	Connected	
Corollary 1.23	$I(A) \in SI(\mathcal{C}(\mathcal{L}))$	$A \subseteq X$ is irreducible	[3, 3.30, 3.31]
Corollary 1.23	$I(A)$ is irreducible in $\mathcal{C}(\mathcal{L})$	$A \subseteq X$ is irreducible	[3, 3.30, 3.31]
Corollary 1.23	$\sqrt{0}$ is irreducible in $\mathcal{C}(\mathcal{L})$	irreducible	[3, 3.30, 3.31]
$SI(\mathcal{C}(\mathcal{L})) \subseteq X$ (1.70)	$\sqrt{x} \in X$	$V(x)$ is irreducible	[2, 4.17, 5.22], [2, 3.27], [3, 3.33], [14, 3.6], [1, 4.28]
$SI(\mathcal{C}(\mathcal{L})) \subseteq X$ (1.70)	$\sqrt{x} \in Min(X)$	$V(x)$ is irreducible component	[2, 5.22], [2, 4.17], [3, 3.27], [3, 3.33], [1, 4.28]
$Max(\mathcal{L}) = X$ and $\mathcal{C}(\mathcal{L})$ satisfies the complete max property (1.67 (2))	$\lvert Max(X)\rvert = 1$	Connected	
Remark 1.66 (5)	1 is finitely constructed	Compact	
Remark 1.66 (5)	1 is countably constructed	Lindelof	

Table 1.2: Examples on the Interplay between topological properties of (X, τ) and algebraic properties of the $X - top$ lattice $\mathcal{L}$

Lemma 1.72 *Let R be a ring and M a top^p-module, that is $\mathcal{L} := LAT(_RM)$ is $Spec^p(M)$-top. Then*

$$SI(\mathcal{C}(LAT(M))) \subseteq Spec^p(M).$$

Proof. Let N be strongly irreducible in $\mathcal{C}(\mathcal{L})$. Suppose that $IK \subseteq N$ for some ideal $I \leq R$ and a submodule $K \leq M$. Then $IK \subseteq P$ for any prime submodule $P \in V(N)$, whence $IM \subseteq P$ or $K \subseteq P$ and so $\sqrt{IM} \subseteq P$ or $\sqrt{K} \subseteq P$, whence $\sqrt{IM} \cap \sqrt{K} \subseteq P$ for all $P \in V(N)$. Since N is radical, $\sqrt{IM} \cap \sqrt{K} \subseteq N$. By assumption, N is strongly irreducible in $\mathcal{C}(\mathcal{L})$, whence $IM \subseteq \sqrt{IM} \subseteq N$ or $K \subseteq \sqrt{K} \subseteq N$. Therefore, $N \in Spec^p(M)$. ∎

Example 1.73 *Let R be a ring and M a top^p-module. By Lemma 1.72, we have $SI(\mathcal{C}(LAT(_RM))) \subseteq Spec^p(M)$. So, all the 1-1 correspondences in Theorem 1.70 hold for this special case. Behboodi and Haddadi proved the second correspondence in [14, Corollary 3.6].*

Example 1.74 *Let R be a ring and $_RM$ a left top^c-module (i.e. $\mathcal{L} = LAT(_RM)$ is X-top, where $X = Spec^c(M)$). If $_RM$ is completely distributive, then $SI(\mathcal{C}(\mathcal{L})) \subseteq X$ by [2, Proposition 5.19] and the 1-1 correspondences of Theorem 1.70 hold. In [2, Proposition 5.22], these correspondences were proved under the additional condition that every coprime submodule of M is strongly irreducible.*

Example 1.75 *Let R be a ring and $_RM$ a left top^s-module (*i.e. *$\mathcal{L} = LAT(_RM)$ is dual X-top, where $X = Spec^s(M)$). By [2, Proposition 4.14], $SH(\mathcal{H}(\mathcal{L})) \subseteq X$*

and so the 1-1 correspondences of Theorem 1.70 hold. These were proved in this special case in [2, Proposition 4.17] under the additional condition that every second submodule of M is strongly hollow.

Example 1.76 *Let R be a ring and ${}_RM$ a left top^{fp}-module (*i.e. $\mathcal{L} = LAT({}_RM)$ *is X-top, where $X = Spec^{fp}(M)$). If ${}_RM$ is duo, then $SI(\mathcal{C}(\mathcal{L})) \subseteq X$ by [3, 3.30] and the 1-1 correspondences of Theorem 1.70 hold. These were also obtained under the same condition in [3, Proposition 3.33].*

Example 1.77 *Let R be a ring and ${}_RM$ a left top^{fc}-module (*i.e. $\mathcal{L} = LAT({}_RM)$ *is dual X-top, where $X = Spec^{fc}(M)$). If ${}_RM$ is duo, then $SH(\mathcal{H}(L)) \subseteq X$ by [4, Proposition 3.25] and Proposition 1.22 and the 1-1 correspondences of Theorem 1.70 hold. These were also obtained under the same condition for this special case in [4, Proposition 3.28].*

Example 1.78 *Let R be a ring and ${}_RM$ a left top^{f}-module (*i.e. $\mathcal{L} = LAT({}_RM)$ *is dual X-top, where $X = Spec^{f}(M)$). If ${}_RM$ has the property that $H(A)$ is first whenever A is irreducible, then $SH(\mathcal{H}(L)) \subseteq X$ and so the 1-1 correspondences of Theorem 1.70 hold. This was proved under the same condition in [1].*

Example 1.79 *Let R be a PID with an infinite number of non-zero prime ideals (e.g. $R = \mathbb{Z}$), $\mathcal{L} := Ideal(R)$, $X = Max(R)$ and consider the topological space* (X, τ).

(1) *$X = V(0)$ is irreducible since 0 is strongly irreducible. However, $0 = \sqrt{0} \notin X$ and so X is not sober by Remark 1.66 (7), whence not spectral.*

(2) *X is T_1 as $Max(X) = X$.*

(3) *X is cofinite: consider a closed set $\emptyset \neq V(I) \subsetneq X$, where $I = (a)$ for some $a \in R \backslash \{0\}$. Since R is a PID, the unique prime factorization of a implies that I is contained in a finite number of primes, i.e. $V(I)$ is finite.*

(4) *X is not regular, not T_2, and not normal. Observe that X is infinite and cofinite, so it does not have disjoint non-empty open sets, although it has disjoint non-empty closed sets.*

Example 1.80 *Let R be a ring, M a left R-module and $X \subseteq LAT({}_RM) \backslash \{M\}$ (resp. $X \subseteq LAT({}_RM) \backslash \{0\}$) and assume that $\mathcal{L} := LAT({}_RM)$ is X-top (resp. dual X-top). If $\mathcal{C}(\mathcal{L})$ is uniform (resp. $\mathcal{H}(\mathcal{L})$ is hollow), then (X, τ) (resp. (X, τ^0)) is connected by Theorem 1.67 (1).*

Example 1.81 *Let R be a commutative domain, $\mathcal{L} := Ideal(R)$, $X \subseteq Ideal(R) \backslash \{R\}$ (resp. $X \subseteq Ideal(R) \backslash \{0\}$), and assume that $\mathcal{L}$ is X-top (resp. $\mathcal{L}$ is dual X-top). If $\sqrt{0} = 0$ (resp. $\sum\limits_{p \in X} p = R$), Then (X, τ) (resp. (X, τ^0)) is connected.*

Example 1.82 *Let R be a UFR with zero devisors. Consider $\mathcal{L} := Ideal(R)$, $X := Spec(R)$ (the prime spectrum of R) and assume that $Min(X)$ is infinite (e.g. $R = Z_n[x]$ with n not prime). Notice that $\sqrt{0} = 0$ (since $Min(X)$ is infinite, if $0 \neq x \in \sqrt{0}$ then $x \in \bigcap\limits_{p \in Min(X)} q$, but this is impossible as R is a UFR).*

- *(X, τ) is connected by Theorem 1.67 (7).*

  Claim: *the intersection of any infinite collection of minimal elements of X is zero. Suppose that $0 \neq I := \bigcap\limits_{q \in \mathbf{m}'} q$ for some infinite subcollection $\mathbf{m}'$*

of $Min(R)$. For any $x \in I\backslash\{0\}$, we have $x = p_1 \cdots p_n$ where $p_1, \cdots, p_n$ are prime elements of R. Notice that $p_1, \cdots, p_n \in I$. For every $q \in \mathbf{m}'$, we have $q = (p_i)$ for some $i \in \{1, 2, \cdots, n\}$, whence $\mathbf{m}'$ is finite (a contradiction).

- *(X, τ) is reducible by Remark 1.66 (1). To prove this, suppose that (X, τ) is irreducible and that $I \cap J = 0$ for some ideals $I, J \leq R$. Then $V(0) = V(I \cap J) = V(I) \cup V(J) = V(\sqrt{I}) \cup V(\sqrt{J}) = V(\sqrt{I} \cap \sqrt{J})$, whence $\sqrt{I} \cap \sqrt{J} = \sqrt{0} = 0$. Since $\sqrt{0} \in SI(\mathcal{C}(\mathcal{L}))$ (by Remark 1.66 (1)), it follows that $I = \sqrt{I} = 0$ or $J = \sqrt{J} = 0$, whence R is a domain, a contradiction.*

Example 1.83 *Let $(G, +)$ be a group and set*

$$L := \{H \mid H \trianglelefteq G \text{ is a normal subgroup of } G\},$$

$$X := \{H \mid H \trianglelefteq G \text{ is a finite normal subgroup of } G\}\backslash\{G\}.$$

Notice that $\mathcal{L} = (L, \cap, +, G, 0)$ is a complete lattice endowed with $\bigvee\limits_{i \in I} N_i := \sum\limits_{i \in I} N_i$ and $\bigwedge\limits_{i \in I} N_i := \bigcap\limits_{i \in I} N_i$.

(1) *$\mathcal{C}(L) = X \cup \{G\}$ as the intersection of any non-empty family of finite normal subgroups is a finite normal subgroup.*

(2) *$SI(\mathcal{C}(\mathcal{L})) \subseteq X$ and so all the 1-1 correspondences of Theorem 1.70 hold.*

(3) *$0 = \sqrt{0} \in X$ and so (X, τ^{cl}) is irreducible and connected (observe that $\overline{\{0\}} = X$ and $\overline{\{0\}}$ is irreducible).*

(4) *$\mathcal{C}(\mathcal{L})$ satisfies the DCC but need not satisfy the ACC (e.g. a p-quasicyclic group [35]).*

(5) *$SI(\mathcal{C}(\mathcal{L})) = X$ if and only if $\mathcal{L}$ is an X-top lattice.*

(6) *If $\mathcal{L}$ is X-top, then the intersection of any nonzero elements in X is nonzero.*

(7) *By Theorem 1.28: (X, τ^{cl}) is $T_1 \Leftrightarrow (X, \tau^{cl})$ is a singleton $\Leftrightarrow (X, \tau^{cl})$ is T_2 $\Leftrightarrow (X, \tau^{cl})$ is discrete.*

(8) *Suppose that $\mathcal{L}$ is an X-top lattice and (X, τ^{cl}) is compact with each element in G having a finite order. Then G is a finite p-group for some prime p. Indeed, since X is compact, by Theorem 1.67 (5), G is finitely constructed. But G is the union of all proper cyclic subgroups, say $G = \sum_{i \in I} H_i$. Then $G = \sum_{j \in F} H_j$ where F is a finite subset of I. Hence G is finite.*

Consequently, the Prüfer group is not X-top (X is the set of all proper subgroups) as it is infinite.

Example 1.84 *Let $(G, +)$ be a group, $Z(G)$ the center of G and set*

$$L := \{H \mid H \trianglelefteq G \text{ is a normal subgroup of } G\},$$

$$X := \{H \mid H \leq Z(G)\} \backslash \{G\}.$$

Notice that $\mathcal{L} = (L, \cap, +, G, 0)$ is a complete lattice with $\bigvee_{i \in I} N_i := \sum_{i \in I} N_i$ and $\bigwedge_{i \in I} N_i := \bigcap_{i \in I} N_i$.

(1) *$\mathcal{C}(L) = X \cup \{G\}$ as the intersection of any non-empty family of subgroups of the center is again in the center.*

(2) *$SI(\mathcal{C}(\mathcal{L})) \subseteq X$ and so all correspondences of Theorem 1.70 hold.*

(3) $0 = \sqrt{0} \in X$ *and so* (X, τ^{cl}) *is irreducible and connected.*

(4) *By Theorem 1.28:* (X, τ^{cl}) *is* $T_1 \Leftrightarrow (X, \tau^{cl})$ *is singleton* $\Leftrightarrow (X, \tau^{cl})$ *is* $T_2 \Leftrightarrow$ (X, τ^{cl}) *is discrete.*

(5) $SI(\mathcal{C}(\mathcal{L})) = X \Leftrightarrow \mathcal{L}$ *is* X*-top. Hence, if* $\mathcal{L}$ *is an* X*-top lattice, then the intersection of any distinct nonzero subgroups in* X *is nonzero.*

(6) *If* G *is finite, then* (X, τ^{cl}) *is spectral by Remark 1.41.*

(7) *Suppose that* $\mathcal{L}$ *is an* X*-top lattice and* (X, τ^{cl}) *is compact with each element in* G *having a finite order. Then* G *is a finite* p*-group for some prime* p*.*

(8) X *is coatomic and* $Z(G)$ *is the unique maximal element of* X*.*

(9) *If* $\mathcal{L}$ *is* X*-top, then* X *is compact as* X *is coatomic and* $Max(X)$ *is finite (by Theorem 1.67 (3)).*

Example 1.85 *Let* $(G, +)$ *be a group,* $Z(G)$ *the center of* G *and set*

$$L := \{H \mid H \trianglelefteq G \text{ is a normal subgroup of } G\},$$

$$X := \{H \mid H \leq Z(G) \text{ is finite }\} \backslash \{G\}.$$

Notice that $\mathcal{L} = (L, \cap, +, G, 0)$ *is a complete lattice with* $\bigvee\limits_{i \in I} N_i := \sum\limits_{i \in I} N_i$ *and* $\bigwedge\limits_{i \in I} N_i := \bigcap\limits_{i \in I} N_i$.

(1) $\mathcal{C}(L) = X \cup \{G\}$ *as the intersection of any non-empty family of finite subgroups of the center is again finite and in the center.*

(2) $SI(\mathcal{C}(\mathcal{L})) \subseteq X$ *and so all correspondences of Theorem 1.70 hold.*

(3) $0 = \sqrt{0} \in X$ *and so* (X, τ^{cl}) *is irreducible and connected.*

(4) *By Theorem 1.28:* (X, τ^{cl}) *is* $T_1 \Leftrightarrow (X, \tau^{cl})$ *is singleton* $\Leftrightarrow (X, \tau^{cl})$ *is* $T_2 \Leftrightarrow (X, \tau^{cl})$ *is discrete.*

(5) $SI(\mathcal{C}(\mathcal{L})) = X \Leftrightarrow \mathcal{L}$ *is* X*-top. Hence, if* $\mathcal{L}$ *is an* X*-top lattice, then the intersection of any distinct nonzero subgroups in* X *is nonzero and so* X *can only be* $\{0\}$ *or a collection of* p*-subgroups for some fixed prime* p*. Otherwise,* $H \in X$ *has order* $p^n q^m l$ *with primes* p *and* q *not dividing* l *and so by the Sylow Theorem [35, Theorem 5.2] there is a Sylow* p*-subgroup* K_1 *of order* p^n *and a Sylow* q*-subgroup* K_2 *of order* q^m*. By Lagrange's Theorem [35, Theorem 1.26], the order of their intersection must divide* p^n *and* q^m *and so the intersection must be zero, whereas* K_1 *and* K_2 *are nonzero elements of* X*. The uniqueness of* p *is clear also by Lagrange's Theorem.*

(6) *If* G *is finite, then* (X, τ^{cl}) *is spectral by Remark 1.41.*

(7) *Suppose that* $\mathcal{L}$ *is an* X*-top lattice and* (X, τ^{cl}) *is compact with each element in* G *having a finite order. Then* G *is a finite* p*-group for some prime* p*.*

(8) X *is coatomic and* $Z(G)$ *is the unique maximal element of* X*.*

(9) *If* $\mathcal{L}$ *is* X*-top, then* X *is compact as* X *is coatomic and* $Max(X)$ *is finite (by Theorem 1.67 (3)).*

CHAPTER 2

REPRESENTATIONS AND COREPRESENTATIONS

Throughout this chapter R is a ring and M is a non-zero R-module. For $N, K \leq M$ and $I \leq R$, we set

$$(K :_R N) := \{a \in R \mid aN \subseteq K\} \text{ and } (N :_M I) := \{x \in M \mid Ix \leq N\}.$$

In particular, we set $Ann(N) := (0 :_R N)$.

2.1 Preliminaries

2.1 *([47, Sec. 41]) We say that an R-submodule $N \leq M$ has a* supplement *K in M iff there is a R-submodule $K \leq M$ minimal with respect to $N + K = M$. The R-module M is said to be* supplemented *iff every R-submodule of M has a supplement in M. We say that $N \leq M$ has* ample supplements *in M [47] iff for* *each submodule* $U \leq M$ *with* $N + U = M$ *there is a supplement* $K \subseteq U$ *of* N *in* M*. The* R*-module* M *is called* amply supplemented *iff every* R*-submodule of* M *has ample supplements in* M*. For example every Artinian module is amply supplemented.*

2.2 *A submodule* $N \leq M$ *is called* small *(or* superfluous*)* in M *[47, 19.1] iff* $N + K \neq M$ *for any* R*-submodule* $K \lneq M$*. An epimorphism of* R*-modules* $f : M \longrightarrow M'$ *is called a* small epimorphism *iff* $Ker(f)$ *is small* M*. An* R*-submodule* $N \leq M$ *is called* large *(or* essential*) [47, 17.1] iff* $N \cap K \neq 0$ *for any* R*-submodule* $0 \neq K \leq M$*. A monomorphism of* R*-modules* $g : M \longrightarrow M'$ *is called a* large monomorphism *iff* $f(M)$ *is large in* M'*.*

2.3 *We say that* M *is a* lifting R*-module [18, 22.2] iff any* R*-submodule* $N \leq M$ *contains a direct summand* $X \leq M$ *such that* N/X *is small in* M/X*. An* R*-module* M *is called* extending *[18, p. 265] iff every nonzero submodule of* M *is essential in a direct summand of* M*.*

2.4 *An* R*-module* M *is called* uniform *[18] iff every nonzero* R*-submodule of* M *is large in* M *(equivalently,* $0 \in LAT(_RM)$ *is irreducible). An* R*-module* M *has uniform dimension* n *[18], and we write* $u.dim(M) = n$*, iff there exists a large monomorphism from a direct sum of* n *uniform* R*-modules to* M*. An* R*-module* M *is* hollow *iff every proper* R*-submodule of* M *is small in* M *(equivalently,* $1 \in LAT(_RM)$ *is hollow). We say that* M *has hollow* dimension n *[18] iff there exists a small epimorphism from* M *to a direct sum of* n *hollow* R*-modules, in this case we write* $h.dim(M) = n$*.*

Lemma 2.5 *([18, Proposition 22.11]) Let M be a nonzero R-module with finite hollow (uniform) dimension.*

(1) *If ${}_RM$ is lifting, then $M = \bigoplus_{i=1}^{n} H_i$ where each H_i is a hollow R-module and $n = h.dim(M)$.*

(2) *If ${}_RM$ is extending, then $M = \bigoplus_{i=1}^{n} U_i$ where each U_i is a uniform R-module and $n = u.dim(M)$.*

Lemma 2.6 *([18, 22.2], [18, 20.34])*

(1) *Every lifting R-module is amply supplemented.*

(2) *The following are equivalent for an amply supplemented R-module M :*

   (a) M has finite hollow dimension.

   (b) M has the DCC on supplements.

   (c) M has the ACC on supplements.

Lemma 2.7 *([47, 41.5, 41.6])*

(1) *If ${}_RM$ is coatomic and every maximal R-submodule of M has a supplement in M, then M is a sum of hollow submodules of M.*

(2) *Let ${}_RM$ be finitely generated. Then M is supplemented if and only if M is a sum of hollow submodules.*

2.2 Primary and Secondary Representations

Till the end of this chapter, the ring R is assumed to be commutative.

2.8 *A proper R-submodule $N \lneq M$ is called* primary *[8] iff whenever $ax \in N$ and $x \notin N$ we have $a^n M \subseteq N$ for some $n \in \mathbb{N}$. If N is a primary submodule of M, then $p := \sqrt{(N :_R M)}$ is prime ideal of R and we say that N is p-*primary. *A submodule $K \leq M$ has a* primary decomposition *[8] iff there are primary submodules $N_1, \cdots, N_n$ of M with $K = \bigcap_{i=1}^{n} N_i$. Such a decomposition of K, if it exists, is called* minimal *iff:*

(1) $\sqrt{(N_i :_R M)} \neq \sqrt{(N_j :_R M)}$ *for* $i \neq j$;

(2) $\bigcap\limits_{i \neq j} N_i \nsubseteq N_j \ \forall j \in \{1, 2, \cdots, n\}$.

Theorem 2.9 *(Lasker-Noether Theorem [8, Theorem 18.20]) Every submodule of a finitely generated module over a Noetherian ring has a primary decomposition.*

Theorem 2.10 *(First uniqueness Theorem of Primary Decompositions [8, Theorem 18.19]) Let R be Noetherian and M an R-module. If $\bigcap\limits_{i=1}^{n} N_i = N = \bigcap\limits_{j=1}^{m} K_j$ are two minimal primary decompositions of $N \leq M$, where N_i is p_i- primary for all $i \in \{1, 2, \cdots, n\}$ and K_j is q_j-primary for all $j \in \{1, 2, \cdots, m\}$, then $n = m$ and $\{p_1, p_2, \cdots, p_n\} = \{q_1, q_2, \cdots, q_m\}$.*

Theorem 2.11 *(Second Uniqueness Theorem of Primary Decomposition ([8, Theorem 18.24]) Let M be a finitely generated module over a Noetherian ring R and $\bigcap\limits_{i=1}^{n} N_i = N = \bigcap\limits_{i=1}^{n} K_i$ be two minimal primary decompositions of $N \leq M$, where N_i*

and K_i are p_i- primary submodules of M for all $i \in \{1, 2, \cdots, n\}$. If p_j is minimal among $\{p_1, p_2, \cdots, p_n\}$ for some $j \in \{1, 2, \cdots, n\}$, then $N_j = K_j$.

2.12 *An R-module M is called* secondary *([30], [32]) iff for any $a \in R$ we have $aM = M$ or $a^n M = 0$ for some $n \in \mathbb{N}$. If M is a secondary R-module, then $p := \sqrt{Ann(M)}$ is a prime ideal of R and M is called* p-secondary. *An R-module M is called* representable *([30], [32]) iff $M = \sum_{i=1}^{n} N_i$, where $N_1, \cdots, N_n$ are secondary R-modules. Moreover, $M = \sum_{i=1}^{n} N_i$ is said to be a* minimal secondary representation *iff $\sqrt{Ann(N_i)} \neq \sqrt{Ann(N_j)}$ whenever $i \neq j$ and $N_j \nsubseteq \sum_{i \neq j} N_i$ for all $j \in \{1, \cdots, n\}$. For each $i \in \{1, 2, \cdots, n\}$, the prime ideal $p_i := \sqrt{Ann(N_i)}$ is called an* attached prime *[32] and we set $Att(M) := \{p_1, \cdots, p_n\}$. A subset $A \subseteq Att(M)$ is called* isolated *iff $q \in A$ whenever $q \in Att(M)$ and $q \subseteq p$ for some $p \in A$.*

2.13 *A prime ideal $p \leq R$ is called a* coassociated prime *[24] to ${}_RM$ iff there is a* hollow *factor M' of M such that $p = \{a \in R \mid aM' \neq M'\}$. The set of coassociated primes of an R-module M is denoted by $Coass(M)$. If ${}_RM$ is representable, then $Att(M) = Coass(M)$ ([45, Theorem 1.14]).*

Theorem 2.14 *([30, Theorem 1]) Every Artinian module is representable.*

Theorem 2.15 *([41, Theorem 2.3]) Every injective module over a Noetherian ring is representable.*

Proposition 2.16 *([32]) Let $M_1, M_2, \cdots, M_n$ be secondary R-submodules of the R-module M. Then $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a p-secondary R-module if and only if M_i is a p-secondary R-submodule of M for all i.*

Theorem 2.17 *(First Uniqueness Theorem of Secondary Representations) ([30, Theorem 2]) If* $\sum_{i=1}^{n} K_i$ *and* $\sum_{j=1}^{m} N_j$ *are two minimal secondary representations for* ${}_RM$, *with* K_i *is* p_i *-secondary for* $i = 1, 2, \cdots, n$ *and* N_j *is* q_j*-secondary for* $j = 1, 2, \cdots, m$, *then* $n = m$ *and* $\{p_1, p_2, \cdots, p_n\} = \{q_1, q_2, \cdots, q_n\}$.

Theorem 2.18 *(Second Uniqueness Theorem of Secondary Representation [13, Theorem 3.2.7]) Let* M *be representable,* $A \subseteq Att(M)$ *an isolated subset and* $M = \sum_{i=1}^{n} K_i$ *a minimal secondary representation for* M *with* K_i *is* p_i*-secondary for* $i \in \{1, \cdots, n\}$. *Then* $\sum_{p_i \in A} K_i$ *is independent of the choice of the minimal second representation.*

Lemma 2.19 *([45, Theorem 1.10]) Every quotient* Q *of a representable module* ${}_RM$ *is representable and* $Att(Q) \subseteq Att(M)$.

Theorem 2.20 *([45, Theorem 1.10]) If* N *is a representable* R*-submodule of the representable module* M, *then* $Att(M/N) \subseteq Att(M) \subseteq Att(N) \cup Att(M/N)$.

Theorem 2.21 *([45, Theorem 1.11]) If* $M_1, \cdots, M_n$ *are representable* R*-modules, then* $\bigoplus_{i=1}^{n} M_i$ *is representable and* $Att(\bigoplus_{i=1}^{n} M_i) = \bigcup_{i=1}^{n} Att(M_i)$.

2.3 Second Representations

Yassemi [43] introduced the notion of *second submodules* of a given non-zero module over a commutative ring. Annin [9] called these *coprime modules* (see also [46]) and used them to dualize the notion of attached primes.

2.22 *A nonzero submodule $K \leq M$ is called* second *[43] iff for any ideal $I \leq R$, we have $IK = 0$ or $IK = K$. The spectrum of second R-submodules of M is denoted by $Spec^s(M)$. If $K \in Spec^s(M)$, then $p := (0 :_R K)$ is a prime ideal, called a* second attached prime *of M and K is called p-*second*. By*

$$Att^s(M) := \{(0 :_R K) \mid K \in Spec^s(M)\} \tag{2.1}$$

we denote the set of second attached primes of M.

Lemma 2.23 *Let $\{K_i\}_{i\in A}$ be family of second R-submodule of M such that $K_j \nsubseteq \sum\limits_{i\in A\setminus\{j\}} K_i$ for all $j \in A$. Let p be a prime ideal of R. Then K_i is p-second for all $i \in A$ if and only if $\sum_{i\in A} K_i$ is p-second.*

Proof. ($\Rightarrow$) Assume that K_i is p-second for all $i \in A$. Clearly, $p = (0 :_R \sum\limits_{i\in A} K_i)$. Let $I \leq R$. If $IK_j = 0$ for some $j \in A$, then $I \subseteq p$, whence $I \sum_{i\in A} K_i = 0$. Otherwise, $IK_j = K_j$ for all $j \in A$ and so $I \sum\limits_{i\in A} K_i = \sum\limits_{i\in A} IK_i = \sum\limits_{i\in A} K_i$. Consequently, $\sum_{i\in A} K_i$ is second.

($\Leftarrow$) Assume that $\sum\limits_{i\in A} K_i$ is p-second and that K_i is p_i-second for $i \in A$. Clearly $p \subseteq p_i$ for all $i \in A$. For any $j \in A$, we have $p_j \sum\limits_{i\in A} K_i = \sum\limits_{i\in A\setminus\{j\}} p_j K_i \subseteq \sum\limits_{i\in A\setminus\{j\}} K_i \neq \sum\limits_{i\in A} K_i$, whence $p_j \sum\limits_{i\in A} K_i = 0$, i.e. $p_j \subseteq p$. Hence $p = p_j$ for all $j \in A$. ∎

Definition 2.24 *We say that an R-module M is (*directly*)* second representable *iff $M = \sum\limits_{i=1}^{n} K_i$ ($M = \bigoplus\limits_{i=1}^{n} K_i$) where K_i is a second R-submodule of M for all $i = 1, 2, \cdots, n$; in this case we call $\sum\limits_{i=1}^{n} K_i$ ($\bigoplus\limits_{i=1}^{n} K_i$) a (*direct*)* second representation *of M. An R-module M is called* semisecond *iff M is a (not necessarily finite) sum of second submodules of M.*

Example 2.25

Let p be a prime number. Any divisible p-group is a semisecond $\mathbb{Z}$-module but not semisimple. This follows from the fact that every divisible p-group is a direst sum of copies of Prüfer group $\mathbb{Z}(p^{\infty})$ which is a 0-second $\mathbb{Z}$-module but not simple (see [47, p. 124], [21, p. 96] for Prüfer group).

$$\mathbb{Z}(p^{\infty}) = \langle g_1,\ g_2,\ g_3,\ ... \mid g_1^p = 1,\ g_2^p = g_1,\ g_3^p = g_2, \cdots \rangle\,. \tag{2.2}$$

2.26 *A (direct) second representation $M = \sum\limits_{i=1}^{n} K_i$ ($M = \bigoplus\limits_{i=1}^{n} K_i$) is called a* minimal *(*direct*)* second representation *for M iff it satisfies the following conditions:*

(1) $(0:_R K_i) \neq (0:_R K_j)$ *for $i \neq j$.*

(2) $K_j \nsubseteq \sum\limits_{i=1, i\neq j}^{n} K_i$ *for all $j = 1, 2, \cdots, n$.*

Let ${}_RM$ be second representable. It is clear that M has a minimal second representation say $\sum\limits_{i=1}^{n} K_i$. Each K_i in such a minimal representation is called a main second submodule of M and $(0:_R K_i)$ is called a main second attached prime *of M. So, the set of main second attached primes is*

$$att^s(M) = \{Ann(K_i) \mid i = 1, \cdots, n\}. \tag{2.3}$$

By Theorem 2.30 below, $att^s(M)$ is independent of the choice of the minimal *second representation of M.*

Theorem 2.27 *(Existence Theorem for Minimal Second Representations) Let M be second representable. Then M has a minimal second representation.*

Proof. The result follows by Lemma 2.23. ∎

2.28 *Every p-second R-module is p-secondary and every (minimal) second representation is a (minimal) secondary representation. So, every second representable R-module is secondary representable and $att^s(M) = Att(M)$. A subset $A \subseteq att^s(M)$ is called* isolated *iff for any $p \in att^s(M)$ with $p \subseteq q$ for some $q \in A$, we have $p \in A$.*

Example 2.29 *The Abelian group $\mathbb{Z}_{18}$ has a minimal secondary representation as a $\mathbb{Z}$-module, namely $\mathbb{Z}_{18} = 2\mathbb{Z}_{18} + 9\mathbb{Z}_{18}$. However, $\mathbb{Z}_{18}$ has* no *second representation ($9\mathbb{Z}_{18}$ is the unique second $\mathbb{Z}$-submodule of $\mathbb{Z}_{18}$).*

In the light of Remark 2.28, we obtain as a direct consequence of Theorem 2.17 and 2.18 the First & Second Uniqueness Theorems for Second Representations:

Theorem 2.30 *(First Uniqueness Theorem of Second Representations) Let M be an R-module with two minimal second representations $\sum_{i=1}^{n} K_i = M = \sum_{j=1}^{m} N_j$, where K_i is p_i-second for all $i \in \{1, 2, \cdots, n\}$ and N_j is q_j-second for all $j \in \{1, 2, \cdots, m\}$. Then $\{p_1, p_2, \cdots, p_n\} = \{q_1, q_2, \cdots, q_m\}$.*

Theorem 2.31 *(Second Uniqueness Theorem of Second Representations) Let M be second representable. If $\sum_{i=1}^{n} K_i = M = \sum_{i=1}^{n} N_i$ are minimal second representa-*

tions for M with K_j and N_j are p_j-second submodules of M and p_j is minimal in $\{p_1, \cdots, p_n\}$ for some $j \in \{1, \cdots, n\}$, then $K_j = N_j$.

Remarks 2.32 *Let M, $M_1, ..., M_n$ be second representable submodules of an R-module L.*

(1) $\sum_{i=1}^{n} M_i$ *is second representable and* $att^s(\sum_{i=1}^{n} M_i) \subseteq \bigcup_{i=1}^{n} att^s(M_i)$.

(2) *Any quotient Q of M is second representable and* $att^s(Q) \subseteq att^s(M)$.

(3) *Let N be second representable submodule of M. Then*

$$att^s(M/N) \subseteq att^s(M) \subseteq att^s(N) \cup att^s(M/N).$$

(4) *If* $M_j \cap \sum_{i \neq j} M_i = 0$ *for all j, then* $\bigoplus_{i=1}^{n} M_i$ *is second representable and*

$$att^s(\bigoplus_{i=1}^{n} M_i) = \bigcup_{i=1}^{n} att^s(M_i).$$

(5) $S^{-1}M$ *is a second representable $S^{-1}R$-module and*

$$att^s(S^{-1}M) = \{p_S \mid p \in att^s(M) \text{ such that } p \cap S = \emptyset\}.$$

Proof. (1) and (5) are clear.

(2) Let $K_1 + \cdots + K_n$ be a minimal second representation for M and $Q = M/N$ for some R-submodule $N \leq M$, then $M/N = \sum_{i=1}^{n}(K_i + N)/N$. It is easy to see that

$(K_i + N)/N$ is second for $i \in \{1, \cdots, n\}$. The result is obtained now by applying (1).

For (3) and (4), apply part (2), Theorem 2.20 and Theorem 2.21. ∎

Proposition 2.33 *Let $M = \sum_{i\in\Lambda} K_i$ (resp. $M = \bigoplus_{i\in\Lambda} K_i$), where K_i is second for every $i \in \Lambda$. If $Att^s(M)$ is finite, then M is second representable (resp. directly second representable).*

Proof. Let $M = \sum_{i\in\Lambda} K_i$ (resp. $M = \bigoplus_{i\in\Lambda} K_i$) such that each K_i is second for every $i \in \Lambda$. Assume that $Att^s(M) = \{p_1, p_2, \cdots, p_n\}$. For $j \in \{1, 2, \cdots, n\}$, set $A_j = \{K_i : i \in \Lambda$ such that $(0 :_R K_i) = p_j\}$. Notice that $N_j = \sum_{K_i \in A_j} K_i$ is second by Proposition 2.23 for each $j \in \{1, 2, \cdots, n\}$. Moreover, $M = \sum_{j=1}^{n} N_j$ (resp. $M = \bigoplus_{j=1}^{n} N_j$). ∎

2.34 *We say that a submodule $K \leq M$ satisfies the* IS-condition *iff for every $I \leq R$ for which $IK \neq 0$, the submodule $IK \leq M$ has a proper supplement in M.*

Remark 2.35 *Let ${}_RM$ be supplemented, $K \leq M$ and $0 \neq H \leq K$. The following conditions are equivalent:*

(1) *K is not contained in any supplement of H in M.*

(2) *H has a proper supplement in M.*

Proof. $(1 \Rightarrow 2)$ Assume that K is not contained in any supplement of H in M. Since M is supplemented, H has a supplement L in M, i.e. $H + L = M$. Indeed $L \neq M$ as $K \nsubseteq L$.

$(2 \Rightarrow 1)$ Assume that H has a proper supplement L in M. Then $K \nsubseteq L$; otherwise, $H + L = L \neq M$. ∎

Lemma 2.36 *Every hollow R-submodule $0 \neq K \leq M$ satisfying the IS-condition is second.*

Proof. *Let $0 \neq K \leq M$ be a hollow R-submodule satisfying the IS-condition is second. Let $I \leq K$ and suppose that $IK \neq 0$. By IS-condition, IK has a proper supplement $L \lneqq M$. It is easy to show that $IK + (L \cap K) = K$. Since K is hollow, $IK = K$ (notice that $L \cap K \neq K$; otherwise, $IK + L = L \neq M$).* ∎

Example 2.37 *The Abelian group $\mathbb{Z}_{18}$, considered as a $\mathbb{Z}$-module, is supplemented but not semisimple. The submodule $K_1 = 9\mathbb{Z}_{18}$ is hollow and satisfies the IS-condition. Notice that $K_2 := 6\mathbb{Z}_{18}$ is hollow and second but does not satisfy the IS-condition (i.e. the IS-condition is not necessary for a hollow submodule module to be second).*

2.38 *We say that an R-module M is (*directly*)* hollow representable *iff M is a finite (direct) sum of hollow R-submodules.*

Proposition 2.39 *Let ${}_RM$ be (directly) hollow representable. If every* maximal *hollow non-zero submodule of M is second, then M is (directly) second representable.*

Proof. Let $M = \sum_{i=1}^{n} H_i$ be a sum of hollow R-submodules. Assume, without loss of generality, that this sum is *irredundant.* **Claim:** $H_1, \cdots, H_n$ are maximal hollow submodules of M. To see this, suppose that H is a hollow submodule of M

with $H_i \leq H$ for some $i \in \{1, \cdots, n\}$ and consider $N := \sum_{j \neq i} H_j$. For any $x \in H$, there are $y \in N$ and $z \in H_i$ such that $x = y + z$. But $z \in H$ implies that $y \in H$. So, $H = (N \cap H) + H_i$. Since H is hollow, either $H \cap N = H$ whence $H \subseteq N$, or $H_i = H$. But $H_i \subseteq H$ and $M = \sum_{i=1}^{n} H_i$ is an irredundant sum, whence $H = H_i$. Hence H_i is maximal hollow. By our assumption, $H_1, \cdots, H_n$ are second, whence $M = \sum_{i=1}^{n} H_i$ is a second representation of M.

If $M = \bigoplus_{i=1}^{n} H_i$ is a direct sum of hollow R-submodules, then one can show similarly that each H_i is a maximal hollow R-submodule of M, whence M is a direct sum of hollow R-submodules. ∎

Example 2.40 *Every Artinian left R-module is hollow representable (see [42, Lemma 3.2]). Let $p \in \mathbb{Z}$ be a prime number. The* Prüfer group *considered as a $\mathbb{Z}$-module, is Artinian and the unique maximal hollow $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty)$ is second.*

Example 2.41 *A lifting R-module M is directly hollow representable if it satisfies any of the following additional conditions:*

(1) *${}_RM$ has a finite hollow dimension [18, Proposition 22.11] (e.g. ${}_RM$ is finitely generated [18, Corollary 22.12]).*

(2) *${}_RM$ has a finite uniform dimension [18, Proposition 22.11] (e.g. ${}_RM$ is finitely cogenerated [18, Corollary 22.12]).*

Inspired by Example 2.41 and Proposition 2.39, we introduce the notion of s-lifting modules.

Definition 2.42 *We call ${}_RM$* s-lifting *iff ${}_RM$ is lifting and every* maximal *hollow submodule of M is second.*

Examples 2.43 (1) *Consider the Abelian group $M = \mathbb{Z}_8$ as a $\mathbb{Z}$-module. Notice that $N = \{0, 4\}$ is the unique second submodule in M, hence M is not second representable. Notice that ${}_{\mathbb{Z}}M$ is Artinian and lifting but not s-lifting.*

(2) *Every semisimple module is s-lifting and trivially semisecond (every simple submodule is second).*

(3) *Every second hollow module is s-lifting but not necessarily simple. Consider the Prüfer group $M = \mathbb{Z}(p^{\infty})$ (see 2.2 that describes the Prüfer group), considered as $\mathbb{Z}$-module. Notice that ${}_{\mathbb{Z}}M$ is not simple. Moreover, ${}_{\mathbb{Z}}M$ is hollow and second whence s-lifting hollow but not semisimple.*

As a direct consequence of Proposition 2.39 and Example 2.41, we obtain the following class of directly second representable modules.

Example 2.44 *If ${}_RM$ is an s-lifting module and has a finite hollow dimension, then M is directly second representable. Clearly, this class is nonempty; e.g. any finite direct sum of Prüfer groups is s-lifting with finite hollow dimension.*

The following example is an s-lifting second module with infinite hollow dimension which is not semisimple.

Example 2.45

Let $\mathbb{P}$ be the set of prime numbers, $A \subseteq \mathbb{P}$ infinite and consider $M := \bigoplus_{p\in A} \mathbb{Z}(p^\infty)$ considered as a $\mathbb{Z}$-module.

Claim: *${}_{\mathbb{Z}}M$ is lifting. This can be obtained by applying [[12], Theorem 2] (the justification is located in the second paragraph of page 60 in [12]). However, we provide here our own proof.*

Let $N \leq M$. Assume, without loss of generality that N is not a direct summand of M (if N is a direct summand of M, then $N/N = 0$ is small in M/N and we are done). Notice that $N = \bigoplus_{p\in A} L_p$, where $L_p \leq \mathbb{Z}(p^\infty)$ for all $p \in A$.

Case 1: *$L_p \neq Z(p^\infty)$ for all $p \in A$. In this case, N is small in M as the set of submodules of $\mathbb{Z}(p^\infty)$ form a chain for all $p \in A$. Indeed, for every $p \in A$: if $L_p + W_p = M$, then $L_p \subseteq W_p = M$.*

Case 2: *$L_p = Z(p^\infty)$ for all $p \in B \subsetneq A$ and $L_p \neq Z(p^\infty)$ $\forall p \notin B$. Let $K = \bigoplus_{p\in B} L_p$. In this case, N/K is small in M/K as the set of submodules of $\mathbb{Z}(p^\infty)$ form a chain for all $p \in A$.*

Notice that the maximal hollow $\mathbb{Z}$-submodules of M are $\{\mathbb{Z}(p^\infty) \mid p \in A\}$ and they are second, whence ${}_{\mathbb{Z}}M$ is s-lifting.

Notice that ${}_{\mathbb{Z}}M$ is second, not semisimple and that $h.\dim({}_{\mathbb{Z}}M) = \infty$.

Example 2.46 *Let $n = p_1 \cdots p_n$ be a product of distinct prime numbers and consider $M = \mathbb{Z}_n[x]$ as a $\mathbb{Z}$-module. Then M is second representable semisimple. Indeed, let $m_j = \frac{n}{p_j}$ for all $j = 1, 2, ..., n$. Set $K_{j_k} = \mathbb{Z}m_j x^k$ for all $j = 1, 2, ..., n$ and $k \in \{0, 1, 2, ...\}$. Then K_{j_k} is simple for all $j = 1, 2, ..., n$ and $k \in \{0, 1, 2, ...\}$ and $K_j = \bigoplus_{k=0}^{\infty} K_{j_k}$ is p_j-second for all $j = 1, 2, ..., n$. Hence $M = \bigoplus_{j=1}^{n} K_j$ is second* *representable while it is semisimple with infinite length.*

The above two examples show also that the finiteness condition on the hollow dimension in Example 2.44 is not necessary.

Example 2.47 *Let $n = p_1 \cdots p_n$ be a product of distinct prime numbers, p any prime number and consider the Abelian group $M = \mathbb{Z}_n[x] \oplus \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. Since $\mathbb{Z}_n[x]$ is second representable (see Example 2.46) and $\mathbb{Z}(p^\infty)$ is second, it follows that ${}_{\mathbb{Z}}M$ is second representable. Notice that ${}_{\mathbb{Z}}M$ is neither semisimple nor second.*

As a direct consequence of Lemma 2.6 and Example 2.44 we obtain:

Corollary 2.48 *Let ${}_RM$ be s-lifting.*

(1) *If M has the ACC on supplements (*e.g.* Noetherian), then M is directly second representable.*

(2) *If M has the DCC on supplements (*e.g.* Artinian), then M is directly second representable.*

Theorem 2.49 *Let M be an R-module.*

(1) *If ${}_RM$ is finitely generated, supplemented and every maximal hollow R-submodule of M is second, then M is second representable.*

(2) *If ${}_RM$ is coatomic, every maximal R-submodule of M has a supplement in M, every maximal hollow R-submodule of M is second and $\mathrm{Att}^s(M)$ is finite, then M is second representable.*

Proof.

(1) Since ${}_RM$ is finitely generated and supplemented, ${}_RM = \sum_{\lambda\in\Lambda} M_\lambda$ a sum of hollow R-submodules by Lemma 2.7 (2). Since ${}_RM$ is finitely generated, this sum can be taken to be finite and it follows that M is second representable by Proposition 2.39.

(2) Since ${}_RM$ is coatomic and every maximal R-submodule of M is has a supplement in M, it follows by Lemma 2.7 (1) that ${}_RM = \sum_{\lambda\in\Lambda} M_\lambda$ a sum of hollow, whence second, R-submodules of M. Since $Att^s(M)$ is finite, it follows by Proposition 2.33 that M is second representable.

∎

Example 2.50 *Theorem 2.49 provides a non-empty class of examples of second representable modules. For example, let p be a prime number and consider the Prüfer group $M = \mathbb{Z}(p^\infty)$, as a $\mathbb{Z}$-module (see (2.2) that describes Prüfer group). Clearly, M is second and supplemented but not finitely generated. This example shows that the finiteness condition of Theorem 2.49 (1) is not necessary.*

Moreover, consider $N = \langle g_k \rangle \leq M$ for some $k \in \mathbb{N}$. Observe that N is finitely generated and supplemented, so by Theorem 2.49, N is second representable if and only if N is second as it is hollow.

2.51 *We define a semisecondary module is one which is a (possible infinite) sum of secondary submodules.*

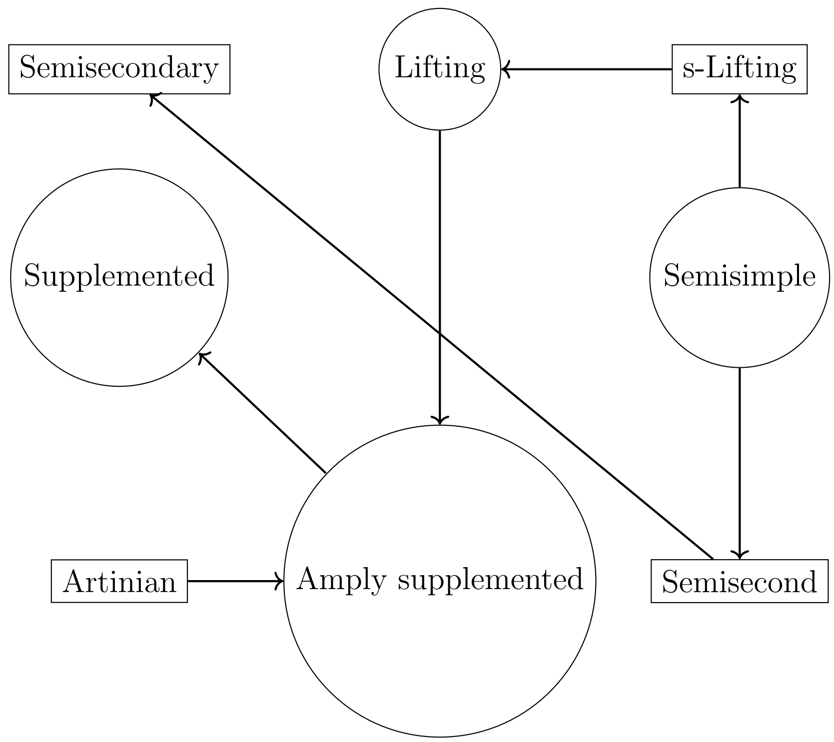

Figure 2.1: s-lifting position chart

Example 2.52 *Let R be a commutative ring with finite prime spectrum (e.g. $R = \mathbb{Z}_n$). Assume that M is coatomic and amply supplemented over R (e.g. an Artinian module over an Artinian ring) in which the maximal hollow submodules are second. Then M is second representable by Theorem 2.49 (2). To show this, let $K \lneq M$ be maximal submodule, whence there is element $x \in M \backslash K$. So, $K + Rx = M$ as K is maximal. Since M is amply supplemented, there is a supplement $N \leq Rx$ of K.*

Example 2.53 *The Abelian group $M = \mathbb{Z}_{12}$, considered as a $\mathbb{Z}$-module, has a secondary representation $M = (4) \oplus (3)$ but no second representation, it has a* *finite hollow dimension (notice that the epimorphism*

$$\phi : M \longrightarrow (4) \oplus (6), \; x \mapsto 2x$$

is small and so h.dim$(M) = 2$). Observe that M is not s-lifting as the submodule (3) *is maximal hollow but not second. This example shows that the assumption that M is s-lifting in Theorem 2.44 cannot be dropped.*

Example 2.54 *Let ${}_RM$ have an infinite number of distinct simple R-submodules $\{S_1, S_2, \cdots\}$ such that $A := \{Ann(S_i) \mid i \in \mathbb{N}\}$ is also infinite. The semisimple module $N = \bigoplus_{i=1}^{\infty} S_i$ is not second representable. This example shows that the finiteness condition on the hollow (uniform) dimension of ${}_RM$ in Theorem 2.44 cannot be dropped.*

Example 2.55 *A multiplication semisimple module $M = \bigoplus_{i=1}^{\infty} S_i$, with infinite number of distinct simple submodules $\{S_i \mid i \in \mathbb{N}\}$ is not second representable. To prove this we claim that $A := \{Ann(S_i) \mid i \in \mathbb{N}\}$ is infinite. Suppose that $Ann(S_i) = P = Ann(S_j)$ for some $i \neq j$. Since ${}_RM$ is multiplication, $S_i = IM$ for some ideal $I \leq R$, whence $I \subseteq Ann(S_j) = P = Ann(S_i)$. But this would mean that $S_i \nsubseteq IM$ (a contradiction). Thus A is infinite as $\{S_i \mid i \in \mathbb{N}\}$ is infinite.*

Proposition 2.56 *Let ${}_RM$ be second representable with a minimal second representation $M = \sum_{i=1}^{n} K_i$ and consider*

$$att^s(M) = \{Ann(K_1), \cdots, Ann(K_n)\};$$
$$Att^s(M) = \{Ann(K) \mid K \in Spec^s(M)\}$$

(1) *$Att^s(M)$ is atomic and $Min(att^s(M)) = Min(Att^s(M))$*

(2) *If there is no small second submodule of M, then $Att^s(M)$ is coatomic and $Max(att^s(M)) = Max(Att^s(M))$.*

Proof.

(1) **Claim:** $\bigcap_{p \in att^s(M)} p \subseteq q$ for every $q \in Att^s(M)$: Let $a \in \bigcap_{p \in att^s(M)} p$. Then $a \in Ann(M)$. It is easy to show that $Ann(M) \subseteq \bigcap_{p \in Att^s(M)} p$, hence $\bigcap_{p \in att^s(M)} p = \bigcap_{p \in Att^s(M)} p$ as $att^s(M) \subseteq Att^s(M)$, whence $\bigcap_{p \in att^s(M)} p \subseteq q$ for all $q \in Att^s(M)$. Now, Suppose that $q \in Min(Att^s(M))$. Then $\bigcap_{p \in att^s(M)} p \subseteq q$. Since $att^s(M)$ is finite and each element in $Att^s(M)$ is prime, it follows that $p \leq q$ for some $p \in att^s(M)$. By the minimality of q in $Att^s(M)$ and $att^s(M) \subseteq Att^s(M)$, we have $p = q$. Therefore, $Min(Att^s(M)) \subseteq Min(att^s(M))$. So, $Att^s(M)$ is atomic.

For the inverse inclusion, let $p \in Min(att^s(M))$. Suppose that $p \notin Min(Att^s(M))$. Then there is $q \in Att^s(M)$ such that $q \subsetneq p$. Since $\bigcap_{p \in att^s(M)} p \subseteq q$ and $att^s(M)$ is finite, $p' \subseteq q$ for some $p' \in att^s(M)$, i.e. $p' \subseteq q \subsetneq p$, which contradicts the minimality of p in $att^s(M)$.

(2) Assume that there is no small second submodule in M.

Claim: For every $p \in Att^s(M)$, we have $pM \neq M$ and $p \subseteq q$ for some $q \in att^s(M)$: Let $p \in Att^s(M)$. Then there is a p-second submodule $K \leq M$. Since K is not small in M, there is a proper submodule $L \lneq M$ such that $K + L = M$ and so $pM = L \neq M$.

Let $p \in Max(Att^s(M))$ and assume, without loss of generality that $pM = \sum_{i=1}^{m} K_i$ with $m \lneq n$ (as $pM \neq M$) and $p \subseteq Ann(K_i)$ for all $i \in \{m+1, m+2, \cdots, n\}$. Since $p \in Max(Att^s(M))$, $n = m+1$ and $p = Ann(K_n)$, i.e. $p \in Max(att^s(M))$. Therefore, $Att^s(M)$ is coatomic and $Max(Att^s(M)) \subseteq Max(att^s(M))$.

For the inverse inclusion, let $q \in Max(att^s(M))$. Suppose that $q \notin Max(Att^s(M))$, so that $q \subsetneq p$ for some $p \in Att^s(M)$. Then $p \subseteq q'$ for some $q' \in att^s(M)$, whence $q \subsetneq p \subseteq q'$, which contradicts the maximality of p in $att^s(M)$. Consequently, $Max(att^s(M)) \subseteq Max(Att^s(M))$.

∎

Example 2.57 *Consider the Abelian group $M = \mathbb{Z}_n$ as a $\mathbb{Z}$-module. We describe the second spectrum of M and find $Att^s(M)$ and $att^s(M)$.*

Example 2.58 *If n is prime, then*

$$Att^s(M) = att^s(M) = \{(n)\}.$$

If n is not prime, then consider the prime factorization $n = \prod_{i=1}^{k} p_i^{n_i}$ and let $m_i :=$

n/p_i for $i \in \{1, 2, \cdots, n\}$. Notice that (m_i) is p_i-second for all $i \in \{1, 2, \cdots, n\}$ and $Att^s(M) = \{(p_1), (p_2), \cdots, (p_k)\}$.

To find $att^s(M)$, we have the following cases:

***Case 1:** $n_i = 1$ for all $i \in \{1, 2, \cdots, n\}$. In this case, $M = \sum_{i=1}^{k}(m_i)$ is a second representation and $att^s(M) = \{(p_1), (p_2), \cdots, (p_k)\}$.*

***Case 2:** $n_j > 1$ for some $j \in \{1, 2, \cdots, n\}$. In this case, M is not second representable since $\sum_{i=1}^{k}(m_i) \subseteq (p_j) \neq M$.*

Example 2.59 *Let M be a second representable $\mathbb{Z}$-module. Then either $0 \in att^s(M)$ or $0 \notin att^s(M)$, and so by Proposition 2.56 we have $Min(Att^s(M)) = \{0\}$ or $Att^s(M) = Min(Att^s(M)) = att^s(M)$. In particular, if M is a torsion module (e.g. $M = \mathbb{Z}_p \times \mathbb{Z}_q$ for some prime numbers p and q), then $Att^s(M) = Min(Att^s(M)) = att^s(M)$.*

Theorem 2.60 *Let ${}_RM$ be Noetherian.*

(1) *Let p be a prime ideal. Then M is p-secondary (p-second) if and only if every nonzero submodule of M is p-secondary (p-second).*

(2) *If $M = \sum_{i=1}^{n} K_i$ is a minimal secondary representation with K_i is p_i-secondary for some prime ideals $\{p_1, \cdots, p_n\} \subseteq Spec(R)$, then $M = \bigoplus_{i=1}^{n} K_i$.*

Proof. For ${}_RM$, consider for every $x \in R$ the endomorphism

$$a_M : M \longrightarrow M,\ x \mapsto ax.$$

(1) We prove the result for the case of p-secondary modules; the case of p-second modules can be proved similarly.

($\Rightarrow$) Let M be a p-secondary module for some prime ideal $p \leq R$. Let $0 \neq K \leq M$. For any $a \notin p$, we have $aM = M$. Since ${}_RM$ is Noetherian, every surjective endomorphism is injective and so a_M is injective. Hence a_M^n is injective for any n, i.e. $a^nK \neq 0$ for all $n \in \mathbb{N}$. On the other hand, $aK \subseteq K = aL$ for some submodule L (as a_M is surjective), whence $K \subseteq L$ and

$$a_K \circ a_L : L \longrightarrow aK$$

is an isomorphism of R-modules. So, hence $L = K$ and $aK = K$. Therefore K is p-secondary. ($\Leftarrow$) trivial.

(2) $M = \sum\limits_{i=1}^{n} K_i$ is a minimal secondary representation with K_i is p_i-secondary for some prime ideals $\{p_1, \cdots, p_n\} \subseteq Spec(R)$. Let $A = \{1, 2, \cdots, n\}$.

Claim: For any $j \in A$, we have $K_j \cap \sum\limits_{i \in A \backslash \{j\}} K_i = 0$. Suppose that $K_j \cap \sum\limits_{i \in A \backslash \{j\}} K_i \neq 0$ for some $j \in A$. Notice that by (1), $N = K_j \cap \sum\limits_{i \in A \backslash \{j\}} K_i$ is p_j-secondary. Set $J := \{m \in A : p_m \nsubseteq p_j\}$. For any $m \in J$, there is $a_m \in p_m \backslash p_j$. Consider $a = \prod\limits_{m \in J} a_m$ and notice that $a \notin p_j$ (as J is finite $a_m \in p_m \backslash p_j$ for all $m \in J$) and so $aN = N$. Suppose that $\sum\limits_{i \in A \backslash \{j\}} x_i \in N$ such that $x_i \in K_i$ for all $i \in A$. Then $a^l \sum\limits_{i \in A \backslash \{j\}} x_i = \sum\limits_{i \in A \backslash (\{j\} \cup J)} x_i$ for some l. But $a^lN = N$ and so $N \subseteq \sum\limits_{i \in A \backslash (\{j\} \cup J)} K_i = \sum\limits_{p_i \subsetneq p_j} K_i$, whence $N = K_j \cap \sum\limits_{p_i \subsetneq p_j} K_i$. Since $N \neq 0$, it follows that $\{i \in A : p_i \subsetneq p_j\} \neq \emptyset$. We have the following cases:

Case 1 : $\{i \in A : p_i \subsetneq p_j\} = \{h\}$. In this case, $N = K_j \cap K_h$, N is p_i-secondary and p_h-secondary at the same time (a contradiction).

Case 2: $\{i \in A : p_i \subsetneq p_j\}$ has more than one element. In this case, $N = K_j \cap \sum_{p_i \subsetneq p_j} K_i$. Let p_h be minimal among all $p_i \subseteq p_j$, $a_h \in p_j \backslash p_h$ and $a_i \in p_i \backslash p_h$ for all $p_h \subsetneq p_i \subsetneq p_j$ if it exists. Consider $b = a_h \prod_{p_h \subsetneq p_i \subsetneq p_j} a_i$. Then $b \in p_i$ for all $p_h \subsetneq p_i \subseteq p_j$ and so $bK_h = K_h$. Since ${}_RM$ is Noetherian, $b^t_{K_h}$ is injective for every t. Hence, for any $\sum_{p_i \subsetneq p_j} x_i \in N$, we have $b^t \sum_{p_i \subsetneq p_j} x_i = b^k x_h$ for some k large enough. But b_N is nilpotent as N is p_j-secondary and $b \in p_j$. So, $x_h = 0$ and $N = K_j \cap \sum_{p_i \subsetneq p_j, p_i \neq p_h} K_i$. Also, the set

$$\{i \in A \mid p_i \subsetneq p_j \text{ and } p_i \neq p_h\} \tag{2.4}$$

has a minimal element as it is finite. We continue removing the minimal elements until we arrive at a set containing exactly one element (i.e. Case 1) which yields a contradiction. Therefore, $N = 0$.

$\blacksquare$

Theorem 2.61 *Let ${}_RM$ be Noetherian and Artinian and assume that for any $p \in Max(R)$ the canonical map $\phi_p : M \longrightarrow M_p$ is injective. Then M is (second) secondary representable if and only if M_p is (second) secondary representable R_p-module for any $p \in Max(R)$.*

Proof. We prove the result for the case of secondary representation; the case of second representation can be proved similarly.

Assume that M_p is a secondary representable R_p-module for any maximal ideal $p \leq R$, say $M_p = \sum_{i=1}^{n} K_i'$ is a minimal secondary representation for M_p where each K_i' is a secondary submodule of M_p and set for all $i \in A = \{1, 2, \cdots, n\}$:

$$\begin{aligned} K^i &:= \{x \in M \mid x/s \in K_i' \text{ for some } s \notin p\}; \\ N^i &:= \{x \in M \mid x/1 \in K_i'\} \end{aligned}$$

Then $N^i = K^i$ and $K_p^i = K_i'$. We may write

$$M_p = K_p^1 + K_p^2 + \cdots + K_p^n.$$

Consider the canonical map:

$$\phi : M \longrightarrow M_p; x \mapsto x/1.$$

Then $\phi^{-1}(K_i') = K^i$ for all $i \in A$ and so $M = \sum_{i=1}^{n} K^i$.

Claim: for any $a \in R$ and all $i \in A$, the map

$$\phi^{a,i} : K^i \longrightarrow K^i; x \mapsto ax$$

is nilpotent or surjective. To show this, suppose that $\phi^{a,i}$ is not surjective. Then $\phi^{a,i}$ is not injective since K^i is Artinian (every injective endomorphism of an Artinian module is surjective). Since $\phi^{a,i}$ is not injective, [11, Proposition 3.9] implies that $\phi_p^{a,i}$ is not injective for some maximal ideal p. So, $\phi_p^{a,i}$ is not surjective (any surjective endomorphism of a Noetherian module is injective). It follows that $\phi_p^{a,i}$ is nilpotent, i.e. for some n we have $a^n x/1 = 0$ for all $x \in K^i$ whence $a^n x = 0$ for all $x \in K^i$ by our assumption.

The converse is clear (see Remark 2.32(5) for second representation case). ▮

Example 2.62 (1) *There exists an R-module M which is p-secondary but not Noetherian, while every submodule of M is p-secondary. Appropriate semisimple modules with infinite lengths provide a source for such modules, see Example 3.38.*

(2) *If $_RM$ is a Noetherian or an Artinian R-module with no zero divisors, then M satisfies the conditions of Theorem 2.61.*

Theorem 2.63 (1) *A multiplication R-module M is semisecond (resp. second representable) if and only if each nonzero submodule of M is semisecond (resp. second representable).*

(2) *A multiplication R-module M which is not hollow is semisecond (resp. second representable) if and only if each non-small proper submodule of M is semisecond (resp. second representable).*

(3) *The following conditions are equivalent for a second representable R-module M with a minimal second representation $\sum_{i=1}^{n} K_i$.*

(a) M is multiplication.

(b) $Att^s(M) = att^s(M) = Min(att^s(M))$ *and every nonzero submodule of* M *has a second representable* $\sum\limits_{j\in A} K_j$ *for some* $A \subseteq \{1, 2, \cdots, n\}$.

(4) *If* M *is semisimple second representable, then the following conditions are equivalent:*

(a) M *is multiplication.*

(b) *The elements of* $att^s(M)$ *are incomparable and every second submodule of* M *is simple.*

(5) *The following conditions are equivalent for an atomic module* ${}_RM$:

(a) M *is semisimple.*

(b) $M = \sum\limits_{i\in A} K_i$ *where every submodule of* K_i *is* p_i*-second for some prime ideal* p_i, $i \in A$.

Proof.

(1) Let ${}_RM$ be multiplication. Let $M = \sum\limits_{i\in A} K_i$ be a semisecond representation of M. Let $0 \neq K \leq M$, whence $K = IM$ for some ideal $I \leq R$. Suppose that $I \nleq p_j$ for all $j \in B \subseteq A$ and $I \leq p_i$ for all $i \in A \backslash B$. Then $IM = \sum\limits_{i\in B} K_j$ and so K is semisecond.

(2) Assume that ${}_RM$ is multiplication and not hollow. Then $M = K_1 + K_2$ such that each of K_1 and K_2 are proper submodules of M and so K_1 and K_2 are not small, whence K_1 and K_2 are semisecond. Therefore M is semisecond.

(3) Let $M = \sum_{i=1}^{n} K_i$ be a minimal second representation.

$(a \Rightarrow b)$ Assume that M is a multiplication module. Let $p \in Att^s(M)$, whence there is $K \leq M$ which is p-second. Since ${}_RM$ is multiplication, $K = IM$ for some ideal $I \leq R$. Without loss of generality, assume that $IM = \sum_{i=1}^{m} K_i$ where $I \subseteq p_i$ for all $i \in \{m+1, m+2, \cdots, n\}$. Since K is p-second, it follows by Proposition 2.23 that $m = 1$ and $p = p_i$ for some $i \in \{1, 2, \cdots, n\}$ and so $Att^s(M) = att^s(M)$. If $K \neq 0$, then there is an ideal $J \leq R$ such that $K = JM = J\sum_{i=1}^{n} K_i = \sum_{i \in A} K_i$, where

$$A = \{i \in \{1, 2, \cdots, n\} \mid J \nsubseteq p_i\}.$$

Assume that $p_i \subsetneq p_j$ and $K_j = IM$, but $IM = I\sum_{i=1}^{n} K_i$. By the minimality of $\sum_{i=1}^{n} K_i$ and using Proposition 2.23, $I \subseteq p_i$ and so $I \subseteq p_j$ (a contradiction).

$(b \Rightarrow a)$ Assume that $K \leq M$ is a nonzero submodule. By our assumption, $K = \sum_{j \in A} K_j$ with some $A \subseteq \{1, 2, \cdots, n\}$. Without loss of generality, assume that $K = \sum_{i=1}^{m} K_i$. Let $I = \bigcap_{i=m+1}^{n} p_i$. Since $I \subseteq p_i$ for all $i \in \{m+1, m+2, .., n\}$ and since every element in $Att^s(M)$ is minimal and strongly irreducible and $Att^s(M)$ is finite, it follows that $I \nsubseteq p_i$ for all $i = 1, 2, \cdots, m$. Therefore, $K = IM$. Consequently, ${}_RM$ is multiplication.

(4) Let ${}_RM$ be semisimple with a second presentation $M = \sum_{i=1}^{n} K_i$. It follows that $Att^s(M) = att^s(M)$. Apply now (3) and observe that every nonzero submodule of M is the second representable module $\sum_{j \in A} K_j$ for some $A \subseteq$

$\{1, 2, \cdots, n\}$ if and only if every second submodule of M is simple.

(5) Assume that M is atomic and semisecond, say $M = \sum_{i \in A} K_i$ where every submodule of K_i is p_i-second for some prime ideal p_i, $i \in A$. For each $i \in \{1, 2, \cdots, n\}$ we set

$$H_i = \{S : S \text{ is simple in } K_i\}. \tag{2.5}$$

Claim: $K_i = \bigoplus_{S \in H_i} S$. If not, then there exits $x \in K_i \setminus \bigoplus_{S \in H_i} S$ and so Rx must contain some $S \in H_i$ and so there is $a \in R$ such that $ax \in S$, whence $S = Rax$. But Rx is p_i-second and $Rax = S \neq 0$, whence $S = Rax = Rx$ and so $x \in S$ (a contradiction). It follows that M is a sum of simple submodules. The converse is trivial.

∎

Example 2.64 *Every semisecond atomic Noetherian R-module is semisimple. This follows directly from Theorem 2.60 (1) and Theorem 2.63 (5).*

Theorem 2.65 *Let E be an injective R-module and $\bigcap_{i=1}^{n} p_i = 0$ where $p_1, p_2, \cdots, p_n$ are incomparable prime ideals.*

(1) *E is second representable and $att^s(E) \subseteq \{p_1, p_2, \cdots, p_n\}$.*

(2) *$Ann(E) = 0$ if and only if $att^s(E) = \{p_1, p_2, \cdots, p_n\}$.*

Proof.

(1) We set $E[I] = (0 :_E I)$. By [41, Lemma 2.1], $E[p_i] = 0$ or $E[p_i]$ is p_i-secondary for all $i \in \{1, 2, \cdots, n\}$. Assume that $E[p_i] \neq 0$.

Claim: $E[p_i]$ is second. Let $I \leq R$. If $I \subseteq p_i$, then $IE[p_i] = 0$. If $I \nsubseteq p_i$, then there is $a \notin p_i$. Since $E[p_i]$ is p_i-secondary, $aE[p_i] = E[p_i]$ and so $IE[p_i] = E[p_i]$.

Notice that $E = \sum\limits_{i=1}^{n} E[p_i]$ follows from [41, Lemma 2.2] as $E = E[0] = E[\bigcap\limits_{i=1}^{n} p_i] = \sum\limits_{i=1}^{n} E[p_i]$. Hence, E is second representable and $att^s(E) \subseteq \{p_1, p_2, \cdots, p_n\}$.

(2) $(\Rightarrow)$ Assume that $Ann(E) = 0$. We want to show that $E = \sum\limits_{i=1}^{n} E[p_i]$ is a minimal second representation. By [41, Lemma 2.2], $E[\bigcap\limits_{i \neq j} p_i] = \sum\limits_{i \neq j} E[p_i]$ for any $j \in \{1, 2, \cdots, n\}$ and so $E \neq \sum\limits_{i \neq j} E[p_i]$ for any $j \in \{1, 2, \cdots, n\}$. Otherwise, $E = E[\bigcap\limits_{i \neq j} p_i] = \sum\limits_{i \neq j} E[p_i]$ for some $j \in \{1, 2, \cdots, n\}$ and so $\bigcap\limits_{i \neq j} p_i$ annihilates every element in E for some $j \in \{1, 2, \cdots, n\}$. But $Ann(E) = 0$, whence $\bigcap_{i \neq j} p_i = 0$, which contradicts the fact that $p_i \nsubseteq p_j$ for all $i \in \{1, 2, \cdots, n\} \backslash \{j\}$. Therefore, $att^s(E) = \{p_1, p_2, \cdots, p_n\}$.

$(\Leftarrow)$ Assume that $att^s(E) = \{p_1, p_2, \cdots, p_n\}$. Suppose that $0 \neq a \in Ann(E)$. Then $a \notin p_j$ for some $j \in \{1, 2, \cdots, n\}$ and so $E[p_j] = 0$ or $aE[p_j] = E[p_j]$. Since p_j is a second attached prime, $E[p_j] \neq 0$. Therefore, $aE = \sum_{i=1}^{n} aE[p_i] \neq 0$.

∎

Example 2.66 *Injective modules over Artinian rings are second representable.*

So, any module over an Artinian ring is embedded in a second representable one, namely, the injective hull of it.

CHAPTER 3

A DUAL NOTION FOR PRIMARY SUBMODULES

3.1 Actions on lattices

Throughout this section R is an arbitrary ring (not necessarily commutative), and M is a nonzero left R-module.

3.1 *Let $(S, \leq_S)$ be a poset and $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. An S-action on $\mathcal{L}$ is a map*

$$\rightharpoonup : S \times L \longrightarrow L \tag{3.1}$$

satisfying the following conditions for all $s_1, s_2 \in S$ and $x, y \in L$:

(1) $s_1 \leq_S s_2 \Rightarrow s_1 \rightharpoonup x \leq s_2 \rightharpoonup x.$

(2) $x \leq y \Rightarrow s \rightharpoonup x \leq s \rightharpoonup y.$

(3) *If $\mathcal{L}$ has a minimum element $0 \in L$, then $s \rightharpoonup 0 = 0.$*

If such action exists, we write $(\mathcal{L}, \rightharpoonup)$ *is a* lattice with an S-action . *We say that* $(\mathcal{L}, \rightharpoonup)$ *is* multiplication *iff for every element* $x \in L$*, there is* $s \in S$ *such that* $x = s \rightharpoonup 1$.

Example 3.2 *Let M be an R-module. The complete lattice of R-submodule $LAT({}_RM)$ has an $Ideal(R)$-action defined by the canonical product IN of an ideal $I \leq R$ with a submodule $N \leq M$.*

Remark 3.3 *Let $(S, \leq_S)$ be a poset and $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action $\rightharpoonup : S \times L \longrightarrow L$. The dual lattice $\mathcal{L}^0$ has an $(S, \geq_S)$-action*

$$\rightharpoonup^0 : S \times L \longrightarrow L,$$

where $(S, \geq_S)$ is the dual poset of $(S, \leq_S)$ (where $s_1 \geq_S s_2 \Leftrightarrow s_2 \leq_S s_1$) and for all $s \in S$ and $x \in L$ we have

$$s \rightharpoonup^0 x = (s \rightharpoonup 1) \vee x \tag{3.2}$$

Definitions 3.4 *Let $(S, \leq_S)$ be a poset and $(\mathcal{L}, \rightharpoonup)$ a complete lattice with an S-action. We say that:*

(1) $x \in L \backslash \{1\}$ *is* preprime *iff for all* $y \in L$ *and* $s \in S$:

$$(s \rightharpoonup 1) \wedge y \leq x \quad \Rightarrow \quad s \rightharpoonup 1 \leq x \quad or \quad y \leq x; \tag{3.3}$$

(2) $x \in L\backslash\{1\}$ *is* prime *iff for all* $y \in L$ *and* $s \in S$ *with*

$$s \rightharpoonup y \leq x \quad \Rightarrow \quad s \rightharpoonup 1 \leq x \quad or \quad y \leq x. \tag{3.4}$$

(3) $x \in L\backslash\{1\}$ *is* coprime *iff for all* $s \in S$:

$$s \rightharpoonup 1 \leq x \quad or \quad (s \rightharpoonup 1) \vee x = 1 \tag{3.5}$$

(4) $x \in L\backslash\{0\}$ *is* second *iff for all* $s \in S$:

$$s \rightharpoonup x = x \quad or \quad s \rightharpoonup x = 0 \tag{3.6}$$

(5) $x \in L\backslash\{0\}$ *is* first *iff for all* $y \in L$ *and* $s \in S$ *with*

$$s \rightharpoonup y = 0 \quad and \quad y \leq x \quad \Rightarrow \quad s \rightharpoonup x = 0 \quad or \quad y = 0. \tag{3.7}$$

The spectrum of preprime elements of $\mathcal{L}$ *(resp. prime, coprime, first) is denoted by* $Spec^{pp}(\mathcal{L})$ *(resp.* $Spec^{p}(\mathcal{L})$, $Spec^{c}(\mathcal{L})$, $Spec^{f}(\mathcal{L})$, $Spec^{s}(\mathcal{L})$*).*

Lemma 3.5 *Let* $(S, \leq_S)$ *be a poset and* $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ *a complete lattice with an* S*-action and define*

$$s \rightharpoonup^* x = (s \rightharpoonup 1) \wedge x \tag{3.8}$$

for all $s \in S$ *and* $x \in L$*. Then* $((\mathcal{L}, \rightharpoonup)^0)^0 = (\mathcal{L}, \rightharpoonup^*)$.

Proof. It is clear that $\rightharpoonup^*$ is an $S-$action on $\mathcal{L}$. For all $s \in S$ and all $p \in L$ we have

$$s \rightharpoonup^{0^0} x = (s \rightharpoonup^0 1^0) \vee^0 p = ((s \rightharpoonup 1) \vee 0) \wedge p = (s \rightharpoonup 1) \wedge p = s \rightharpoonup^* p. \tag{3.9}$$

$\blacksquare$

Remarks 3.6 *Let $(S, \leq_S)$ be a poset and $(\mathcal{L}, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action.*

(1) *0 is prime if and only if 1 is first.*

(2) *$SH(\mathcal{L}) \subseteq Spec^p(\mathcal{L}^0)$.*

(3) *If $(\mathcal{L}, \rightharpoonup)$ is multiplication, then $SH(\mathcal{L}) = Spec^p(\mathcal{L}^0)$.*

Assume that $(\mathcal{L}, \rightharpoonup)$ is multiplication. Let $x \in Spec^p(\mathcal{L}^0)$. Suppose that $x \leq y \vee z$ for some $y, z \in L$. Since $(\mathcal{L}, \rightharpoonup)$ is multiplication, $y = s \rightharpoonup 1$ for some $s \in S$, and so $x \leq (s \rightharpoonup 1) \vee z$, i.e. $s \rightharpoonup^0 z \leq^0 x$. Since $x \in Spec^p(\mathcal{L}^0)$, we have $s \rightharpoonup^0 1^0 \leq^0 x$ or $z \leq^0 x$ and so $x \leq (s \rightharpoonup 1)$ or $x \leq \vee z$. So, $Spec^p(\mathcal{L}^0) \subseteq SH(\mathcal{L})$. The inverse inclusion follows by (2).

(4) *$x \in L \backslash \{1\}$ is coprime in $(\mathcal{L}, \rightharpoonup)$ if and only if x is coprime in $(\mathcal{L}, \rightharpoonup^*)$.*

(5) *$x \in L \backslash \{0\}$ is first if and only if 0 is prime in $[0, x]$.*

($\Rightarrow$) Let $x \in L \backslash \{0\}$ is first. Observe that the maximum element in the sublattice $[0, x]$ is x. Suppose that $s \rightharpoonup y = 0$ for some $y \leq x$. Since x is first, $y = 0$ or $s \rightharpoonup x = 0$. So 0 is prime in $[0, x]$.

($\Leftarrow$) *Let* 0 *be prime in* $[0,x]$. *Suppose that* $s \rightharpoonup y = 0$ *with* $y \leq x$. *Since* $y \in [0,x]$, $y = 0$ *or* $s \rightharpoonup x = 0$ *as* x *is the maximum element of* $[0,x]$.

(6) $x \in L\backslash\{0\}$ *is second if and only if* 0 *is coprime in the interval* $[0,x]$.

(7) $x \in L\backslash\{1\}$ *is prime in* $(\mathcal{L}, \rightharpoonup^*)$ *if and only if* x *is preprime in* $(\mathcal{L}, \rightharpoonup)$.

3.7 *Many results in the literature for prime, coprime, second, first, and other types of spectra of submodules of a module can be generalized to a lattice with an* S*-action. For example, if* $(\mathcal{L}, \rightharpoonup)$ *is multiplication, then* $\mathcal{L}$ *is* $Spec^p(\mathcal{L})$*-top as we have for all* $s \in S$ *and* $p \in L$:

$$V(s \rightharpoonup p) \subseteq V(s \rightharpoonup 1) \cup V(p) \subseteq V(s \rightharpoonup 1 \wedge p) \subseteq V(s \rightharpoonup p).$$

For the special case of modules over a ring, see [34, Theorem 3.5].

Definition 3.8 *Let* $\mathcal{L} = (L, \wedge, \vee)$ *be a lattice. Let* $x, y, z \in L$, *with* $x \leq y$ *and* $x \leq z$. *We define* $y \sim z$ *iff for all* $y' \leq y$, *there exists* $z' \leq z$ *such that* $y' \vee x = z' \vee x$, *and for all* $z' \leq z$, *there exists* $y' \leq y$ *such that* $y' \vee x = z' \vee x$. *It is clear that* $\sim$ *is an equivalence relation. Denote the equivalence class of* $y \geq x$ *by* y/x, *and define*

$$L/x := \{y/x \mid y \in L \text{ and } x \leq y\}.$$

Define $y/x \leq^q z/x$ *iff for all* $y' \leq y$, *there exists* $z' \leq z$ *such that* $y' \vee x = z' \vee x$.

Define the meet $\wedge^q$ *and the join* $\vee^q$ *on* L/x *such that*

$$y/x \wedge^q z/x := (y \wedge z)/x \text{ and } y/x \vee^q z/x := (y \vee z)/x.$$

If $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ *is complete, then the meet* $\wedge^q$ *and the join* $\vee^q$ *on* L/x *is defined as*

$$\bigwedge_{i\in A}^{q}(x_i/x) = (\bigwedge_{i\in A} x_i)/x \text{ and } \bigvee_{i\in A}^{q}(x_i/x) = (\bigvee_{i\in A} x_i)/x). \tag{3.10}$$

We define for an element $x \in L$*, the* quotient lattice $\mathcal{L}/x = (L/x, \wedge^q, \vee^q)$.

Remark 3.9 *Let* $(S, \leq_S)$ *be a poset and* $(\mathcal{L}, \rightharpoonup)$ *a lattice with an* S*-action. Define for all* $s \in S$ *and* $\forall y/x \in \mathcal{L}/x$*:*

$$s \rightharpoonup^q y/x = (s \rightharpoonup y) \vee x \tag{3.11}$$

Then $(\mathcal{L}/x, \rightharpoonup^q)$ *is a lattice with an* S*-action.*

Theorem 3.10 *Let* $(S, \leq_S)$ *be a poset and* $(\mathcal{L}, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ *a complete lattice with an* S*-action.*

(1) $Spec^c(\mathcal{L}) = Spec^s(\mathcal{L}^0)$.

(2) $Spec^c(\mathcal{L}^0) = Spec^s(\mathcal{L}^*)$.

(3) *If* $x \in L\backslash\{1\}$ *is prime, then*

$$Spec^f(\mathcal{L}/x) = (\mathcal{L}/x)\backslash\{x/x\}.$$

(4) *Assume that the following additional two conditions are satisfied for our action:*

$$s \rightharpoonup (y \vee z) = s \rightharpoonup y \vee s \rightharpoonup z \text{ for all } s \in S \text{ and } y, z \in L; \quad (3.12)$$

$$s \rightharpoonup y \leq y \text{ for all } s \in S \text{ and } y \in L. \quad (3.13)$$

Then $x \in L \backslash \{1\}$ *is prime* $\Leftrightarrow Spec^f(\mathcal{L}/x) = (\mathcal{L}/x) \backslash \{x/x\}$.

Proof.

(1) $p \in Spec^c(\mathcal{L}) \Leftrightarrow s \rightharpoonup 1 \leq p$ or $(s \rightharpoonup 1) \vee p = 1$.

$\Leftrightarrow s \rightharpoonup 1 \vee p = p$ or $s \rightharpoonup^0 p = 0^0$.

$\Leftrightarrow s \rightharpoonup^0 p = p$ or $s \rightharpoonup^0 p = 0^0$.

$\Leftrightarrow p \in Spec^s(\mathcal{L}^0)$.

(2) $p \in Spec^c(\mathcal{L}^0) \Leftrightarrow s \rightharpoonup^0 1^0 \leq^0 p$ or $(s \rightharpoonup^0 1^0) \vee^0 p = 1^0$.

$\Leftrightarrow (s \rightharpoonup 1) \vee 0 \geq p$ or $((s \rightharpoonup 1) \vee 0) \wedge p = 0$.

$\Leftrightarrow (s \rightharpoonup 1) \wedge p = p$ or $(s \rightharpoonup 1) \wedge p = 0$.

$\Leftrightarrow s \rightharpoonup^* p = p$ or $s \rightharpoonup^* p = 0$.

$\Leftrightarrow p \in Spec^s(\mathcal{L}^*)$.

(3) Let $x \in L \backslash \{1\}$ be prime. **Claim:** $y/x \in \mathcal{L}/x$ is first.

Let $s \rightharpoonup^q z/x = x/x$ and $z/x \leq^q y/x$ and suppose that $z/x \nleq x/x$. Then $((s \rightharpoonup z) \vee x)/x = x/x$. It follows that $((s \rightharpoonup z) \vee x) = x$, and hence $((s \rightharpoonup z) \leq x$. Since x is prime, $((s \rightharpoonup 1) \leq x$ or $z \leq x$. But $z \leq x$

implies that $z = x$, and so $z/x = x/x$. Therefore, $((s \rightharpoonup 1) \leq x$, and so $(s \rightharpoonup 1) \vee x = x$. Hence $s \rightharpoonup^q 1/x = x/x$. Therefore, $s \rightharpoonup^q y/x = x/x$.

(4) Assume that the additional conditions (3.12) and (3.13) are satisfied and that $Spec^f(\mathcal{L}/x) = (\mathcal{L}/x)\backslash\{x/x\}$. **Claim:** x is prime in $\mathcal{L}$.

Suppose that $s \rightharpoonup y \leq x$ and $y \nleq x$. Since $s \rightharpoonup y \leq x$, we have $(s \rightharpoonup y) \vee x = x$. It follows by (3.12) that $s \rightharpoonup (y \vee x) = s \rightharpoonup y \vee s \rightharpoonup x$. However, (3.13) implies that $s \rightharpoonup x \leq x$, hence

$$s \rightharpoonup (y \vee x) = s \rightharpoonup y \vee s \rightharpoonup x \leq (s \rightharpoonup y) \vee x = x.$$

Therefore $(s \rightharpoonup (y \vee x) \vee x)/x = x/x$, whence $s \rightharpoonup^q (y \vee x)/x = x/x$. But $1/x$ is first in $\mathcal{L}/x$, whence $(y \vee x)/x = x/x$ or $s \rightharpoonup^q 1/x = x/x$. Notice that $(y \vee x)/x = x/x$ cannot happen as $y \nleq x$. Thus $s \rightharpoonup^q 1/x = x/x$. Whence $s \rightharpoonup 1 \vee x = x$, i.e. $s \rightharpoonup 1 \leq x$.

∎

Remark 3.11 *Let $(S, \leq_S)$ be a poset and $(\mathcal{L}, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action. Since $Spec^c(\mathcal{L}) = Spec^s(\mathcal{L}^0)$ by 3.10 (2), the result on the second spectrum can be dualized to the coprime spectrum.*

3.2 PS-hollow representation

Throughout this Section, R is a commutative ring with unity and M is an R-module.

3.12 *One can dualize the notion of primary submodules of ${}_RM$ by considering primary submodules in the dual lattice $\mathcal{L}^0 := LAT(M)^0$ with the dual action $\rightharpoonup^0$ (3.2) of the poset $Ideal(R)$: an R-submodule $N \leq M$ is a* dual primary submodule *of M iff whenever $N \leq IM + L$ for some ideal $I \leq R$ and some submodule $L \leq M$, we have $N \leq L$ or $N \leq aM$ for all $a \in I$. Results on such submodules can be obtained by directly dualizing the results on primary submodules.*

In this section, we consider a sort of dual notion of that of a *primary submodule* of ${}_RM$ which is weaker than the *dual primary submodule*. We are taking the exact dual of the preprime submodule which we defined in (3.3). Recall that, the preprime elements in $(LAT(M), \rightharpoonup)$ are exactly the prime elements in $(LAT(M), \rightharpoonup^*)$, where $I \rightharpoonup N = IN$ is the canonical action of an ideal $I \leq R$ on a submodule $N \leq M$.

Definition 3.13 *We say that an R-submodule $N \leq M$ is* pseudo strongly hollow submodule *(or* PS-hollow *for short) iff for any ideal $I \leq R$ and any R-submodule $L \leq M$, we have*

$$N \subseteq IM + L \Rightarrow N \subseteq IM \text{ or } N \subseteq L. \tag{3.14}$$

We say that ${}_RM$ is a pseudo hollow module *(or PS-hollow for short) iff M is a PS-hollow submodule of itself, that is, M is PS-hollow iff for any ideal $I \leq R$ and any R-submodule $L \leq M$, we have*

$$M = IM + L \Rightarrow M = IM \text{ or } M = L. \tag{3.15}$$

Example 3.14 *Let ${}_RM$ be second. Every R-submodule $N \leq M$ is a PS-hollow submodule of M. Indeed, suppose that $N \subseteq IM + L$ for some $L \leq M$ and $I \leq R$. Since ${}_RM$ is second, either $IM = 0$ whence $N \subseteq L$, or $IM = M$ whence $N \subseteq IM$. In particular, every second module is a PS-hollow module.*

Remark 3.15 *It is clear that any strongly hollow submodule of M is PS-hollow; the converse holds in case ${}_RM$ is multiplication.*

Example 3.16

(1) *There exists an R-module M which is not multiplication but all of its PS-hollow submodules are strongly hollow. Consider the module M from Example 2.43 (3). Notice that ${}_RM$ is not a multiplication module, however every R-submodule of M is PS-hollow (even strongly hollow).*

(2) *A PS-hollow submodule $N \leq M$ need not be hollow. Consider $M = \mathbb{Z}_2[x]$ as a $\mathbb{Z}$-module. Set $N := x\mathbb{Z}_2[x]$, and $L := (x+1)\mathbb{Z}_2[x]$. Then $N, L \lneqq M$ and $M = L + N$ is PS-hollow which is not hollow. Indeed, $x^i = x^{i-1}(x+1) - x^{i-2}(x)$ for all $i \geq 2$ and $1 = (x+1) - x$. On the other hand, observe that $IM = M$ or $IM = 0$ for every $I \leq \mathbb{Z}$.*

Lemma 3.17 *Let $N \leq M$ is a PS-hollow submodule. If I is minimal in the set $A := \{I \leq R \mid N \subseteq IM\}$, then I is a hollow ideal of R.*

Proof. Let $I = (J + K)$ for some ideals $J, K \leq R$. Notice that $N \subseteq IM = (J+K)M = JM + KM$, whence $N \subseteq JM$ or $N \subseteq KM$, i.e. $J \in A$ or $K \in A$. By the minimality of I, it follows that $J = I$ or $K = I$. Therefore, I is hollow. ∎

3.18 *Let $N \leq M$ be a PS-hollow submodule and set*

$$A_N := \{I \leq R \mid N \subseteq IM\},\ H_N := Min(A) \text{ and } In(N) := \bigcap_{I \in H_N} IM. \tag{3.16}$$

Notice that A_N is non-empty as $R \in A$, while H_N might be empty and in this case $In(N) = M$ (however $H_N \neq \emptyset$ if R is Artinian). When N is clear from the context, we drop it from the index of the above notations. We say that N is an H-PS-hollow submodule *of M. Every element in H is called an* associated hollow ideal of M. *We write $Ass^h(M)$ to denote the set of all associated hollow ideals of M.*

Proposition 3.19 *Let R be an Artinian ring, N and L be incomparable PS-hollow submodules of M and $H \subseteq Ass^h(M)$. Then $N + L$ is H-PS-hollow if and only if N and L are H-PS-hollow.*

Proof. $(\Rightarrow)$ Let $N \leq M$ and $L \leq M$ be H-PS-hollow submodules.

Claim 1: $H_{N+L} = H_N = H$.

Consider $I \in H_N = H_L$. Clearly, $I \in A_{N+L}$. If $I \notin H_{N+L} := Min(A_{N+L})$, then there is $I' \subsetneq I$ such $N \subseteq N + L \subseteq I'M$ which contradicts the minimality of I in A_N.

Conversely, let $I \in H_{N+L}$. Clearly, $I \in A_N \cap A_L$. If $I \notin H_N$, then there is $I' \in H_N = H_L$ with $I' \subseteq I$ and therefore $N + L \subseteq I'M$, whence $I = I'$ since $I' \in A_{N+L}$. Therefore, $H_{N+L} = H_N = H$.

Claim 2: $N + L$ is PS-hollow.

Suppose that $N + L \subseteq JM + K$ for some ideal $J \leq R$ and a submodule $K \leq M$. Then $N \subseteq N + L \subseteq JM + K$ and so $N \subseteq JM$ or $N \subseteq K$. Similarly $L \subseteq N + L \subseteq JM + K$ and so $L \subseteq JM$ or $L \subseteq K$. Suppose that $N \subseteq JM$, whence there is $I \in H$ such that $N \subseteq IM$ and $I \subseteq J$ (as R is Artinian) and so $L \subseteq IM \subseteq JM$. Therefore, either $N + L \subseteq JM$ or $N + L \subseteq K$. Hence $N + L$ is PS-hollow.

($\Leftarrow$) Assume that $N+L$ is H-PS-hollow. It is clear that $H_{N+L} \subseteq H_L$. Assume that $L \subseteq IM$. Then $N+L \subseteq IM+L$ and $N+L \nsubseteq L$ as N and L are incomparable, whence $N + L \subseteq IM$ and so $H_L \subseteq H_{N+L}$. Therefore, $H_L = H_{N+L}$. Similarly, $H_N = H_{N+L}$. ▮

3.20 *We say that a module M is* PS-hollow representable *iff M can be written as a* finite *sum of PS-hollow submodules. A module M is called* directly PS-hollow representable *(or DPS-hollow representable, for short) iff M is a* finite direct *sum of PS-hollow submodules. A module M is called* semi-pseudo strongly hollow representable *(or* SPS-hollow representable*, for short) iff M is a sum of PS-hollow submodules. We call $M = \sum_{i=1}^{n} N_i$, where each N_i is H_i-PS-hollow, a* minimal PS-hollow representation *for M iff the following conditions are satisfied:*

(1) $H_1, H_2, \cdots, H_n$ *are distinct.*

(2) $N_j \nsubseteq \sum_{i=1, i\neq j}^{n} N_i$ *for all* $j \in \{1, 2, \cdots, n\}$.

If such a minimal PS-hollow representation for M exists, then we call each N_i a main PS-hollow submodule of M *and the elements of $H_1, H_2, \cdots, H_n$ are called* *the* main associated hollow ideals *of* M; *the set of the main associated hollow ideals of* M *is dented by* $ass^h(M)$.

Proposition 3.21 *(Existence of minimal PS-hollow representation) If* R *is an Artinian ring, then every PS-hollow representable* R*-module has a minimal PS-hollow representation.*

Proof. Let $M = \sum_{i \in A} K_i$, where A is finite and K_i is an H_i-PS-hollow submodule $\forall i \in A$.

Step 1: Remove the redundant submodules $K_j \subseteq \sum_{i \neq j} K_i$. This is possible by the finiteness of A.

Step 2: Gather all submodules K_m that share the same H to construct an H-PS-hollow $N \leq M$ as a sum of such H-PS-hollow submodules (this is possible by Proposition 3.19). ∎

Remark 3.22 *Let* R *be Artinian and* $N \leq M$ *be an* H*-PS-hollow submodule. If* $In(N)$ *is PS-hollow, then* $In(N)$ *is* H*-PS-hollow. To show this, observe that for any ideal* $I \leq R$*, we have* $N \subseteq IM$ *if and only if there exists* $I' \in H$ *such that* $N \subseteq I'M$ *with* $I' \subseteq I$ *(as* R *is Artinian), whence* $In(N) \subseteq IM$ *if and only if* $N \subseteq IM$.

Lemma 3.23 *Let* R *be Artinian,* $N \leq M$ *be and* H*-PS-hollow and* $In(N) \leq L$ *whenever* $N \leq L \leq M$*. Then* $In(N)$ *is* H*-PS-hollow.*

Proof. Let $K = In(N) := \bigcap_{I \in H} IM$. Suppose that $K \subseteq JM + L$ for some $J \leq R$ and $L \leq M$. If $K \nsubseteq JM$, then $N \nsubseteq JM$ and so $N \subseteq L$, whence $K \subseteq L$. Therefore K is PS-hollow. Thus, by the Remark 3.22, $In(N)$ is H-PS-hollow. ∎

Example 3.24 *If R is Artinian, then every multiplication R-module M satisfies the conditions of Lemma 3.23 and so $In(N)$ is H-PS-hollow for every H-PS-hollow $N \leq M$ (in fact, $In(N) = N$ in this case).*

Remark 3.25 *Let R be Artinian and M a multiplication R-module. It is easy to see that there is a unique minimal PS-hollow representation of M up to the order,* i.e. *if $\sum_{i=1}^{n} N_i = M = \sum_{j=1}^{m} K_j$ are two minimal PS-representations such that each N_i is H_i-PS-hollow and each K_j is H'_j-PS-hollow, then $n = m$ and $\{N_1, \cdots, N_n\} = \{K_1, \cdots, K_n\}$.*

Theorem 3.26 *(First uniqueness theorem of PS-hollow representation) Let R be Artinian and $\sum_{i=1}^{n} N_i = M = \sum_{j=1}^{m} K_j$ be two minimal PS-representations for ${}_RM$ such that N_i is H_i-PS-hollow for each $i \in \{1, \cdots, n\}$ and K_j is H'_j-PS-hollow for each $j \in \{1, \cdots, m\}$. Then $n = m$, $\{H_1, H_2, \cdots, H_n\} = \{H'_1, H'_2, \cdots, H'_n\}$ and $In(N_i) = In(K_j)$ whenever $H_i = H'_j$.*

Proof. Set $N'_i = In(N_i)$ and $K'_j = In(K_j)$ for $i \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, m\}$.

Claim: For any $i \in \{1, 2, \cdots, n\}$, there is $j \in \{1, 2, \cdots, m\}$ such that $N'_i = K'_j$.

Step 1: Suppose that there exists some $i \in \{1, 2, \cdots, n\}$ for which $N_i \nsubseteq K'_j$ for all $j \in \{1, 2, \cdots, m\}$. Then for any $j \in \{1, 2, \cdots, m\}$, there is $J'_j \in H'_j$ such that $N_i \nsubseteq J'_jM$. But $N_i \subseteq M = \sum_{j=1}^{n} K_j \subseteq \sum_{j=1}^{m} J'_jM$, whence $N_i \subseteq J'_jM$ for some j (a contradiction). So, $N_i \subseteq K'_j$ for some $j \in \{1, 2, \cdots, m\}$.

Step 2: We show that $N'_i \subseteq K'_j$.

Since $N_i \subseteq K'_j$, we have $N_i \subseteq IM$ for all $I \in H'_j$. Since R is Artinian, there is a minimal ideal $J_I \leq I$ such that $N_i \subseteq J_I M$ and so

$$N'_i = In(N_i) = \bigcap_{I \in H_i} IM \subseteq \bigcap_{I \in H'_j} J_I M \subseteq K'_j.$$

Similarly, for any $j \in \{1, 2, \cdots, m\}$, there is some $i \in \{1, 2, \cdots, n\}$ such that $K'_j \subseteq N'_i$. Therefore, for any $i \in \{1, 2, \cdots, n\}$, there is some $j \in \{1, 2, \cdots, m\}$ such that $N'_i = K'_j$ as $N'_1, N'_2, \cdots, N'_n$ are incomparable.

Claim: $H_i = H'_j$ whenever $N'_i = K'_j$.

Let $N'_i = K'_j$. Pick any $I \in H_i$. Then $N_i \subseteq IM$, whence $K'_j = N'_i \subseteq IM$. Since R is Artinian, there is a minimal ideal $I' \in H'_j$ such that $I' \leq I$, and therefore $I' = I$ as I is minimal with respect to $N_i \subseteq IM$. Hence $H_i \subseteq H'_j$. One can prove similarly that $H'_j \subseteq H_i$. So, $H_i = H'_j$. ∎

Theorem 3.27 *(Second uniqueness theorem of PS-hollow representation) Let R be Artinian, M be an R-module with two minimal PS-hollow representations $\sum_{i=1}^{n} N_i = M = \sum_{j=1}^{n} K_j$ with N_i is H_i-PS-hollow for each $i \in \{1, \cdots, n\}$ and K_j is H_j-PS-hollow for each $j \in \{1, \cdots, n\}$. If H_m is minimal in $\{H_1, H_2, \cdots, H_n\}$, then either $N_m = K_m$ or $In(N_m)$ is not PS-hollow.*

Proof. Let H_m be minimal in $\{H_1, H_2, \cdots, H_n\}$ such that $In(N_m)$ is PS-hollow. For any $j \neq m$, there is $I_j \in H_j \backslash H_m$. But $\sum_{j \neq m} I_j M + N_m = M$ and so $In(N_m) \subseteq \sum_{j \neq m} I_j M + N_m$. Since $I_j \in H_j \backslash H_m$, it follows that $In(N_m) \nsubseteq I_j M$ for all $j \in \{1, \cdots, n\} \backslash \{m\}$ and so $In(N_m) \subseteq N_m$, whence $In(N_m) = N_m$. One can prove similarly that $In(K_m) = K_m$. It follows that

$$N_m = In(N_m) \overset{\text{Theorem 3.26}}{=} In(K_m) = K_m.$$

$\blacksquare$

Corollary 3.28 *Let R be Artinian and $\sum_{i=1}^{n} N_i = M = \sum_{i=1}^{n} K_i$ be two minimal PS-hollow representations of ${}_RM$ such that N_i is H_i-PS-hollow for $i \in \{1, 2, \cdots, n\}$ and K_i is H_i-PS-hollow for $i \in \{1, 2, \cdots, n\}$. If $In(N)$ is PS-hollow whenever N is a main PS-hollow submodule of M, then $N_i = K_i$ for all $i \in \{1, 2, \cdots, n\}$.*

Proof. Apply Theorem 3.27 and observe that H_i is minimal in $\{H_1, H_2, \cdots, H_n\}$ for each $i \in \{1, 2, \cdots, n\}$ as $In(N_i)$ is PS-hollow: otherwise, $H_j \subsetneq H_i$ for some $i \neq j$ and $In(N_j)$ can replace $N_i + N_j$ whence $\sum_{i=1}^{n} N_i$ is not minimal (a contradiction).

$\blacksquare$

3.29 *We say that an R-module M is* pseudo distributive *iff for all $L, N \leq M$ and every $I \leq R$ we have*

$$L \cap (IM + N) = (L \cap IM) + (L \cap N). \tag{3.17}$$

Every distributive R-module is indeed pseudo distributive. The two notions coincide for multiplication modules.

Example 3.30 *A pseudo distributive module need not be distributive. Consider $M := \mathbb{Z}_2[x]$ as a $\mathbb{Z}$-module. Let $N := xM$, $L := (x+1)M$ and $K = \mathbb{Z}_2$. Then*

$N, L, K \leq M$ are R-submodules and

$$(K \cap L) + (K \cap N) = 0 \neq K = K \cap (L + N).$$

Notice that M is pseudo distributive as $IM = 0$ or $IM = M$ for every $I \leq R$.

Remark 3.31 *Assume that M is a (directly) hollow representable module for which every maximal hollow is PS-hollow. Then M is (directly) PS-hollow representable.*

Proposition 3.32 *(1) If ${}_RM$ is pseudo distributive, then every hollow submodule of M is PS-hollow.*

(2) If ${}_RM$ is s-lifting, then every maximal hollow submodule of M is PS-hollow.

Proof.

(1) Let M is pseudo distributive. Let $N \leq M$ be hollow. Suppose that $N \subseteq IM + L$, whence $N = (IM + L) \cap N = IM \cap N + L \cap N$ as M is pseudo distributive. Since N is hollow, $N = IM \cap N$ or $N = L \cap N$, therefore $N \subseteq IM$ or $N \subseteq L$. So, N is PS-hollow.

(2) Let ${}_RM$ be s-lifting. Suppose that $K \leq M$ is a maximal hollow submodule of M and that $K \leq IM + L$. Since M is s-lifting, there exists $K' \subseteq K$ and $N \leq M$ such that $K' \oplus N = M$ and $K \backslash K'$ is small in $M \backslash K'$.

Case 1: $K' = 0$: i.e. $M = N$. Since K is second, we have $K = IK \subseteq IN = IM$.

Case 2: $K' \neq 0$: We claim that $K = K'$. To prove this, let $x \in K$. Then there are $y \in K'$ and $z \in N$ such that $x = y + z$. But $y \in K$, whence $z \in K$. Therefore, $K \subseteq K' \oplus (K \cap N)$, but K hollow implies that $K = K'$ or $K = K \cap N$. But $K' \neq 0$, whence $K = K'$; otherwise, $K' \cap N \neq 0$. Therefore, $M = K \oplus N$. Now, it is easy to show that

$$IM + L \leq (IM \cap K + L \cap K) \oplus (IM \cap N + L \cap N),$$

and so

$$K \leq (IM \cap K + L \cap K) \oplus (IM \cap N + L \cap N),$$

whence $K \leq IM \cap K + L \cap K$. Since $IM \cap K + L \cap K \leq K$, it follows that $K = IM \cap K + L \cap K$ and so $K = IM \cap K$ or $K = L \cap K$ which implies that $K \leq IM$ or $K \leq L$.

∎

Examples 3.33 (1) *Every (directly) hollow representable pseudo distributive module is (directly) PS-hollow representable.*

(2) *Every s-lifting with finite hollow dimension is directly PS-hollow representable.*

(3) *The $\mathbb{Z}$-module $M = \mathbb{Z}_n$ is PS-hollow representable. To see this, consider the prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$, and set $n_i = \frac{n}{p_i^{m_i}}$ for $i \in \{1, \cdots, k\}$. Then $M = \sum_{i=1}^{k} (n_i)$ is a minimal PS-hollow representation for M, and (n_i) is H_i-PS-hollow where $H_i = \{(n_i)\}$ for $i \in \{1, 2, \cdots, k\}$.*

(4) *The $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$ is PS-hollow representable ($M = 4Z_{12} + 3Z_{12}$), but M is not second representable. Observe that M is not semisimple and is even not s-lifting as $3Z_{12} \leq \mathbb{Z}_{12}$ is a maximal hollow $\mathbb{Z}$-subsemimodule but not second.*

(5) *Any Noetherian semisimple R-module is directly PS-hollow representable.*

(6) *Any Artinian semisimple R-module is directly PS-hollow representable.*

Lemma 3.34 *Let $N \leq M$ be an H-PS-hollow submodule such that every non-small submodule $K \leq N$ can be written as IM for some ideal $I \leq R$. Every non-small submodule $K \leq N$ is H-PS-hollow submodule and $K \subseteq IM$ if and only if $N \subseteq IM$.*

Proof. Let $K \leq N$ be a non-small submodule. Suppose that $K \subseteq IM + L$ and $K \nsubseteq L$. Then $N \nsubseteq L$. Since K is not small, there is a proper submodule K' of N such that $K + K' = N$. Since $K \subseteq IM + L$, it follows that $N = K + K' \subseteq IM + L + K'$. Suppose that $N \subseteq L + K'$. Since K' not small in N, we have $K' = JM$ for some $J \leq R$, and therefore $N \subseteq K'$ (a contradiction). Hence $N \subseteq IM$ and so $K \subseteq IM$, whence K is PS-hollow.

Claim: $H_K = H$. Let $K \subseteq IM$ for some $I \leq R$. Then $N = K+K' \subseteq IM+K'$. Since N is PS-hollow and $K' \neq N$, we have $N \subseteq IM$. Therefore, $K \subseteq IM$ if and only if $N \subseteq IM$. ∎

Example 3.35 *Consider $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}$-module. Then $K_1 = 3\mathbb{Z}_{12}$ and $K_2 = 4\mathbb{Z}_{12}$ satisfy the assumptions of Lemma 3.34. Notice that ${}_{\mathbb{Z}}M$ is not semisimple.*

3.36 *A module ${}_RM$ is called* comultiplication *[4] iff for every submodule $K \leq M$, we have $K = (0 :_M (0 :_R K))$.*

Theorem 3.37 *Let ${}_RM$ be semisimple, B the set of all maximal second submodules of M, and assume that $Ann(M) \neq \bigcap_{K \in B \setminus \{N\}} Ann(K)$ for any $N \in B$. The following conditions are equivalent:*

(1) *Every second submodule of M is simple.*

(2) *Every PS-hollow submodule of M is simple.*

(3) *${}_RM$ is multiplication.*

(4) *${}_RM$ is comultiplication.*

Proof. Let $M = \bigoplus_{S \in A} S$, where S is a simple submodule of M for all $S \in A$.

($3 \Rightarrow 2$) Assume that ${}_RM$ is multiplication. Let $N \leq M$ be H-PS-hollow. Suppose that N is not simple. Then N contains properly a simple submodule $S' \in A$. Since S' is not small in N, Lemma 3.34 implies that S' is H-PS-hollow. But there is another simple submodule S'' of N (as N is not simple). Let $J = Ann(S'')$. It follows that $S' \subseteq JM$ while $N \nsubseteq JM$ (which contradicts Lemma 3.34). Hence ${}_RN$ a is simple submodule.

($2 \Rightarrow 1$) Assume that every PS-hollow submodule of M is simple. Let N be the second submodule $\bigoplus_{i \in A} S_i$, and consider $N \subseteq IM + L$.

Case 1: $I \subseteq Ann(N)$. In this case, $N \cap IM = 0$. Since $N \subseteq IM + L$ and N is PS-hollow, we have $N \subseteq L$.

Case 2: $I \nsubseteq Ann(N)$. In this case, $N = IN \subseteq IM$. Therefore, N is PS-hollow and hence simple.

$(1 \Rightarrow 3)$ Assume that every second submodule of M is simple. Consider a submodule $K = \bigoplus\limits_{S \in C \subseteq A} S$ of M and set $I := \bigcap\limits_{S \in A \backslash C} Ann(S)$. Notice that $K = IM$, otherwise, $I \subseteq Ann(S)$ for some $S \in C$ whence $Ann(M) = \bigcap_{S \in A \backslash \{S\}} Ann(S)$ (a contradiction).

$(3 \Rightarrow 1)$ Let ${}_RM$ be multiplication. Suppose that $K \leq M$ is a second submodule which is not simple. Since ${}_RM$ is multiplication, $K = IM$ for some $I \leq R$. Since K is not simple, there is simple submodule $S \subsetneq K$; say $S = JM$ for some $J \leq R$. Notice that J annihilates another simple submodule of K, but all simple submodules of K have the same annihilator namely $Ann(K)$ and so $JM \cap S = 0$ (a contradiction).

$(1 \Rightarrow 4)$ Assume that every second submodule of M is simple. Consider a submodule $K = \bigoplus\limits_{S \in C \subseteq A} S$ of M and set $I := (0 :_R K)$. Suppose that $(0 :_M I) \neq K$, whence there is a simple submodule $S' \leq M$ with $S' \cap K = 0$ and $I \subseteq Ann(S')$ which is not allowed by our assumption as it would yields $Ann(M) = \bigcap\limits_{S \in B} Ann(S) = \bigcap\limits_{S \in B \backslash \{S'\}} Ann(S)$ (a contradiction to the assumption).

$(4 \Rightarrow 1)$ Let ${}_RM$ be comultiplication. Let $K \leq M$ be second. For any simple $S \leq K$ we have

$$K = (0 :_M (0 :_R K)) = (0 :_M (0 :_R S)) = S, \tag{3.18}$$

i.e. ${}_RK$ is simple. ∎

Example 3.38 *Consider the $\mathbb{Z}$-module $M = \prod_{i=1}^{\infty} \mathbb{Z}_{p_i p_i'}$, where p_i and p_i' are primes and $p_i \neq p_j$, $p_i' \neq p_j'$ for all $i \neq j \in \mathbb{N}$ and $p_i' \neq p_j$ for any i and j. Let the simple $\mathbb{Z}$-modules K_{p_i} and $K_{p_i'}$ be such that $(0 : K_{p_i}) = (p_i)$ and $(0 : K_{p_i'}) = (p_i')$, so*

$$M = \bigoplus_{i=1}^{\infty} K_{p_i} \oplus \bigoplus_{i=1}^{\infty} K_{p_i'}.$$

Every second $\mathbb{Z}$-submodule of M is simple, while ${}_{\mathbb{Z}}M$ is not multiplication. Notice that the assumption on $Ann(M)$ in Theorem 3.37 is not satisfied for this $\mathbb{Z}$-module, which shows that this condition cannot be dropped.

Corollary 3.39 *If ${}_R M$ is semisimple second representable with $aat^s(M) = Min(aat^s(M))$, then the following conditions are equivalent:*

(1) *Every second submodule of M is simple.*

(2) *Every PS-hollow submodule of M is simple.*

(3) *M is multiplication.*

(4) *M is comultiplication.*

Proof. Since M is second representable, the set B defined in Theorem 3.37 is finite. Since $Ann(S_i)$ is prime for every $i \in A$ and $aat^s(M) = Min(aat^s(M))$ (i.e. different annihilators of simple submodules of M are incomparable), we have $Ann(M) \neq \bigcap_{K \in B \setminus \{N\}} Ann(K)$ for every $N \in B$. The result follows now from Theorem 3.37. ∎

Example 3.40 *Consider $M = \mathbb{Z}_{30}[x]$ as a $\mathbb{Z}$-module. Let $K_i = (10x^i)$, $N_i = (15x^i)$ and $L_i = (6x^i)$. Set $K := \bigoplus_{i=1}^{\infty} K_i$, $N := \bigoplus_{i=1}^{\infty} N_i$ and $L := \bigoplus_{i=1}^{\infty} L_i$. Notice that*

$$M = K + N + L.$$

It is clear that M is second representable semisimple with infinite length, and

$$Att^s(M) = att^s(M) = Min(att^s(M)) = \{(2), (3), (5)\}.$$

Since K is second but not simple, ${}_{\mathbb{Z}}M$ is not comultiplication (by Theorem 3.37 (notice also that ${}_{\mathbb{Z}}M$ is not multiplication).

Example 3.41 *Consider $M = \mathbb{Z}_{30} = (10) + (6) + (15)$. It is clear that M is a second representable, multiplication, comultiplication and semisimple $\mathbb{Z}$-module in which $aat^s(M) = Min(aat^s(M))$ and every second submodule of M is simple. By Corollary 3.39, every PS-hollow submodule of M is simple, and so $(10), (6)$ and (15) are the only PS-hollow submodules of M.*

Theorem 3.42 (1) *If $M = \sum_{i=1}^{n} K_i$ is a minimal second representation of M with $aat^s(M) = Min(aat^s(M))$ and $K_i \cap \sum_{j \neq i} K_j$ is PS-hollow in M for all $i \in \{1, 2, \cdots, n\}$, then $M = \bigoplus_{i=1}^{n} K_i$ if and only if $K_i \cap K_j = 0$ for all $i \neq j$.*

(2) *Let ${}_R M$ be distributive and $M = \sum_{i=1}^{n} K_i$ be a minimal PS-hollow representation such that every submodule of K_i is zero or strongly irreducible or H_i-PS-hollow. Then $M = \bigoplus_{i=1}^{n} K_i$.*

Proof.

(1) Assume that $K_i \cap K_j = 0$ for all $i \neq j$ in $\in \{1, 2, \cdots, n\}$. Set $I_i = \bigcap\limits_{j \neq i} Ann(K_i)$. Since $aat^s(M) = Min(aat^s(M))$, we have $I_i M = K_i$. Also, $K_i \cap \sum\limits_{j \neq i} K_j \subseteq K_i$. Since $K_i \cap \sum\limits_{j \neq i} K_j$ is PS-hollow and each $K_j = I_j M$ for all $j \neq i$, we have $K_i \cap \sum\limits_{j \neq i} K_j \subseteq \sum\limits_{j \neq i} K_j$ implies that $K_i \cap \sum\limits_{j \neq i} K_j \subseteq K_l$ for some $l \neq i$, whence $K_i \cap \sum\limits_{j \neq i} K_j \subseteq K_l \cap K_i = 0$.

(2) Since ${}_RM$ is distributive, it is enough to prove that $K_i \cap K_j = 0$ for all $i \neq j$ in $\{1, 2, \cdots, n\}$. Suppose that $K_i \cap K_j \neq 0$ for some $i \neq j$. But $0 \neq K_i \cap K_j \subseteq K_i$, whence $K_i \cap K_j$ is strongly irreducible or H_i-PS-hollow. Suppose that $K_i \cap K_j$ is strongly irreducible. Since $K_i \cap K_j \subseteq K_i \cap K_j$, it follows that $K_i \subseteq K_i \cap K_j$ or $K_j \subseteq K_i \cap K_j$ and so $K_i \subseteq K_j$ or $K_j \subseteq K_i$ which contradicts the minimality of $\sum_{i=1}^n K_i$. So, $K_i \cap K_j$ is H_i-PS-hollow and at the same time H_j-PS-hollow, which contradicts the minimality of $\sum\limits_{i=1}^{n} K_i$. Therefore $K_i \cap K_j = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$.

∎

Examples 3.43 (1) *Every second representable semisimple module satisfies the assumptions of Theorem 3.42 (2).*

(2) $M = \mathbb{Z}_n$, *considered as a* $\mathbb{Z}$*-module,* M *satisfies all assumptions of Theorem 3.42 ((1) and (2)).*

Theorem 3.44 *Let* R *be Artinian and* $M = \sum\limits_{i=1}^{n} K_i$ *be a minimal PS-hollow representation of* ${}_RM$*. Suppose that the submodules of* K_i *are PS-hollow* $\forall i \in$

$\{1, 2, \cdots, n\}$. *If* $In(K_i) \cap In(K_j) = 0$ $\forall i \neq j$ *in* $\{1, \cdots, n\}$, *then* $M = \bigoplus_{i=1}^{n} K_i$.

Proof. Assume that $In(K_i) \cap In(K_j) = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$. For each $j \in \{1, 2, \cdots, n\}$, set $N_j := K_j \cap \sum_{i \neq j} K_i$. Then $N_j \subseteq In(K_i)$ for some $i \neq j$. Otherwise, $N_j \nsubseteq In(K_i)$ for all $i \neq j$, and so for all $i \neq j$ there is $I_i \in H_i$ such that $N_j \nsubseteq I_i M$. But $N_j \subseteq \sum_{i \neq j} K_i \subseteq I_i M$ and N_j is a PS-hollow submodule by assumption, whence $N_j \subseteq I_i M$ for some $i \neq j$ in $\{1, \cdots, n\}$ (a contradiction).

Observe that $N_j \subseteq K_j \subseteq In(K_j)$ and so $N_j \subseteq In(K_i) \cap In(K_j)$ for some $i \neq j$ in $\{1, \cdots, n\}$. It follows that $N_j = 0$ for all $j \in \{1, 2, \cdots, n\}$ and therefore $M = \bigoplus_{i=1}^{n} K_i$. ∎

Corollary 3.45 *Let* R *be Artinian and* $M = \sum_{i=1}^{n} K_i$ *a minimal PS-hollow representation of* ${}_RM$. *Suppose that the nonzero submodules of* $In(K_i)$ *are* H_i*-PS-hollow for all* $i \in \{1, 2, \cdots, n\}$, *where* K_i *is* H_i*-PS-hollow for each* $i \in \{1, 2, \cdots, n\}$. *Then* $M = \bigoplus_{i=1}^{n} K_i$.

Proof. Suppose that $In(K_i) \cap In(K_j) \neq 0$ for some $i \neq j$ in $\{1, \cdots, n\}$. Then $In(K_i) \cap In(K_j)$ is H_i-PS-hollow, and at the same time $In(K_i) \cap In(K_j)$ is H_j-PS-hollow, which is a contradiction since $H_i \neq H_j$ as $M = \sum_{i=1}^{n} K_i$ is a minimal PS-hollow representation. Therefore $In(K_i) \cap In(K_j) = 0$. The result is obtained by Theorem 3.44. ∎

REFERENCES

[1] J. Abuhlail and Ch. Lomp, *On topological lattices and an application to module theory*, J. Algebra Appl., 15, Article Number: 1650046 (2016).

[2] J. Abuhlail, *Zariski topologies for coprime and second submodules*, Algebra Colloq., 22, 47-72 (2015).

[3] J. Abuhlail, *A Zariski topology for modules*, Commun. Algebra 39, 4163-4182 (2011).

[4] J. Abuhlail, *A dual Zariski topology for modules*, Topology Appl., 158 (3) (2011) 457-467.

[5] J. Abuhlail, *A Zariski Topology for Bicomodules and Corings*, Appl. Categ. Structures, 16, 13-28 (2008).

[6] J. Abuhlail, *Fully coprime comodules and fully coprime corings*, Appl. Categ. Structures, 14, 379-409 (2006).

[7] J. Abuhlail and Ch. Lomp, *On the notion of strong irreducibility and its dual*, J. Algebra Appl., 12, Article Number: 1350012 (2013).

[8] A. Altman and S. Kleiman, *A Term of Commutative Algebra*, MIT (2012).

[9] S. Annin, *Associated and Attached Primes over Noncommutative Rings,* Ph.D. Dissertation, University of California at Berkeley (2002).

[10] H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh, *On the prime spectrum of a module and Zariski topologies*, Commun. Algebra 38, 4461-4475 (2010).

[11] M. Atiyah and I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. (1969).

[12] Y. Baba and M. Harada, *On Almost M-Projectives and Almost M-Injectives*, Tsukuba J. Math. 14, 53-69 (1990).

[13] M. Baig, *Primary Decomposition and Secondary Representation of Modules over a Commutative Ring*, Thesis, Georgia State University (2009).

[14] M. Behboodi and M. R. Haddadi, *Classical Zariski topology of modules and spectral spaces*, Int. Elec. J. Algebra 4, 104-130 (2008).

[15] M. Behboodi and M. R. M. Haddadi, *Classical Zariski topology of modules and spectral spaces, II,* Int. Elec. J. Algebra 4, 131-148 (2008).

[16] N. Bourbaki, *General Topology, Part I*, Addison-Wesley (1966).

[17] N. Bourbaki, *Algebra. Chapitre 8*, Springer-Verlag Berlin Heidelberg, (2012).

[18] J. Clark, Ch. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules, Supplements and Projectivity in Module Theory,* (Frontiers in Mathematics), Birkhäuser (2006).

[19] P. M. Cohn, *Basic Algebra: Groups, Rings and Fields*, Springer London (2003).

[20] J. Dauns, *Prime modules*, J. Rein. Ang. Math. 298, 165-181 (1978).

[21] Laszlo Fuchs. *Infinite abelian groups*. Academic Press, INC. New York 1973.

[22] Steven Galovich, *Unique Factorization Rings with Zero Divisors*, Mathematics Magazine, Vol. 51, 276-283 (1978).

[23] V. Camillo, *Distributive Modules*, J. Algebra, 36, 16-25 (1975).

[24] L. Chambles, *Coprimary decomposition, N-dimention and divisibility. Application to Artinian modules,* Commun. Algebra 9, 1131-1146 (1981).

[25] G. Gratzer, *Lattice Theory: Foundations*, Birkhäuser (2010).

[26] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142, 43-60 (1969).

[27] T. Hungerford, *Algebra,* Graduate Texts in Mathematics 73, Springer (1974).

[28] T. Inoue, *Sum of hollow modules*, Osaka J. Math. 20, 331-336 (1983).

[29] N. Jacobson, *Basic algebra I,* Freeman, San Francisko (1974).

[30] D. Kirby, *Coprimary decomposition of Artinian modules*, J. London Math. Soc. 6, 571-576 (1973).

[31] L. Lang, *Algebra*, Springer (2000).

[32] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symp. Math. 11, 23-43 (1973).

[33] R. L. McCasland and P. F. Smith, *Zariski spaces of modules over arbitrary rings*, Commun. Algebra 34, 3961-3973 (2006).

[34] R. McCasland, M. Moore and P. Smith, *On the spectrum of a module over a commutative ring*, Commun. Algebra 25, 79-103 (1997).

[35] J. S. Milne, *Group Theory*, University of Michigan-Ann Arbor (2013).

[36] M. Mohammed and A. Ghafur Bin Ahmad, *On some properties of hollow and hollow dimension modules*, Comm. Pure Appl. Math., 2, 156-161 (2013).

[37] J. Munkres, *Topology: A First Course*, second edition, Prentice Hall, Upper Saddle River, NJ07458. (2000).

[38] O. Nil, K. T. Derya, *Hollow dimension of modules*, J. Zhejiang Univ. Sci., 6, 1055-1057 (2005).

[39] A. Parkash and A. K. Maloo, *Distributive and multiplication modules*, Beitr. Algebra Geom. 52, 405–412 (2011).

[40] I. R. Shafarevich, *Basic Notions of Algebra*, Springer-Verlag Berlin Heidelberg (2005).

[41] R. Y. Sharp, *Secondary representations for injective modules over commutative Noetherian rings*, Proc. Edinb. Math. Soc. 20, 143-151 (1976).

[42] Semra Tekin, *Modules with Coprimary Decompositions*, Izmir Institute of Technology, M.Sc. Thesis (2009).

[43] S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno) 37, 273-278 (2001).

[44] S. Yassemi, *Coassociated primes of modules over a commutative ring,* Math. Scand. 80, 175-187 (1997).

[45] S. Yassemi, *Coassociated primes*, Commun. Algebra 23, 1473-1498 (1995).

[46] I. Wijayanti, *Coprime modules and comodules*, Ph.D. Dissertation, Heinrich-Heine Universität, Düsseldorf (2006).

[47] R. Wisbauer, *Foundations of Module and Ring Theory, A Handbook for Study and Research*, CRC Press, (1991).

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Index